1-5: Least-squares I

- $A : k \times n$. Usually
$$k > n$$
otherwise easily the minimum is zero.
- Analytical solution:
$$\begin{aligned} f(x) =& (Ax-b)^T(Ax-b) \\ =& x^TA^TAx - 2b^TAx + b^Tb \end{aligned}$$
$$\nabla f(x) = 2A^TAx - 2A^Tb = 0$$

1-5: Least-squares II

- Regularization, weights:

$$\frac{1}{2}\lambda x^T x + w_1(Ax - b)_1^2 + \cdots + w_k(Ax - b)_k^2$$

2-4: Convex Combination and Convex Hull I

- Convex hull is convex

$$
\begin{aligned}
x &= \theta_1 x_1 + \cdots + \theta_k x_k \\
\bar{x} &= \bar{\theta}_1 \bar{x}_1 + \cdots + \bar{\theta}_{\bar{k}} \bar{x}_{\bar{k}}
\end{aligned}
$$

Then

$$
\begin{aligned}
& \alpha x + (1-\alpha)\bar{x} \\
=& \alpha\theta_1 x_1 + \cdots + \alpha\theta_k x_k + \\
& (1-\alpha)\bar{\theta}_1 \bar{x}_1 + \cdots + (1-\alpha)\bar{\theta}_{\bar{k}} \bar{x}_{\bar{k}}
\end{aligned}
$$

2-4: Convex Combination and Convex Hull II

Each coefficient is nonnegative and

$$\begin{aligned} & \alpha\theta_1 + \cdots + \alpha\theta_k + (1-\alpha)\bar{\theta}_1 + \cdots + (1-\alpha)\bar{\theta}_{\bar{k}} \\ = \; & \alpha + (1-\alpha) = 1 \end{aligned}$$

2-7: Euclidean Balls and Ellipsoid I

We prove that any

$$x = x_c + Au \text{ with } \|u\|_2 \leq 1$$

satisfies

$$(x - x_c)^T P^{-1} (x - x_c) \leq 1$$

Let

$$A = P^{1/2}$$

because P is symmetric positive definite.
Then

$$u^T A^T P^{-1} Au = u^T P^{1/2} P^{-1} P^{1/2} u \leq 1.$$

2-10: Positive Semidefinite Cone I

- $\mathbf{S}_{+}^{n}$ is a convex cone. Let

$$X_1, X_2 \in S_{+}^{n}$$

For any $\theta_1 \geq 0, \theta_2 \geq 0$,

$$z^T(\theta_1 X_1 + \theta_2 X_2)z = \theta_1 z^T X_1 z + \theta_2 z^T X_2 z \geq 0$$

2-10: Positive Semidefinite Cone II

- Example:

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$$

  is equivalent to

$$x \geq 0, z \geq 0, xz - y^2 \geq 0$$

- If $x > 0$ or $(z > 0)$ is fixed, we can see that

$$z \geq \frac{y^2}{x}$$

  has a parabolic shape

2-12: Interaction I

- When t is fixed,

$$\{(x_1, x_2) \mid -1 \leq x_1 \cos t + x_2 \cos 2t \leq 1\}$$

gives a region between two parallel lines

This region is convex

2-13: Affine Function I

- $f(S)$ is convex:

  Let

$$f(x_1) \in f(S), f(x_2) \in f(S)$$

$$\begin{aligned} & \alpha f(x_1) + (1-\alpha) f(x_2) \\ = & \alpha(Ax_1 + b) + (1-\alpha)(Ax_2 + b) \\ = & A(\alpha x_1 + (1-\alpha)x_2) + b \\ \in & f(S) \end{aligned}$$

2-13: Affine Function II

- $f^{-1}(C)$ convex:

$$x_1, x_2 \in f^{-1}(C)$$

  means that

$$Ax_1 + b \in C, Ax_2 + b \in C$$

  Because C is convex,

$$\begin{aligned} &\alpha(Ax_1 + b) + (1-\alpha)(Ax_2 + b) \\ =&A(\alpha x_1 + (1-\alpha)x_2) + b \in C \end{aligned}$$

  Thus

$$\alpha x_1 + (1-\alpha)x_2 \in f^{-1}(C)$$

2-13: Affine Function III

- Scaling:

$$\alpha S = \{\alpha x \mid x \in S\}$$

- Translation

$$S + a = \{x + a \mid x \in S\}$$

- Projection

$$T = \{x_1 \in R^m \mid (x_1, x_2) \in S, x_2 \in R^n\}, S \subseteq R^m \times R^n$$

- Scaling, translation, and projection are all affine functions

2-13: Affine Function IV

For example, for projection

$$f(x) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0$$

I: identity matrix

- Solution set of linear matrix inequality

$$C = \{S \mid S \preceq 0\} \text{ is convex}$$
$$f(x) = x_1 A_1 + \cdots + x_m A_m - B = Ax + b$$
$$f^{-1}(C) = \{x \mid f(x) \preceq 0\} \text{ is convex}$$

2-13: Affine Function V

But this isn't rigourous because of some problems in arguing

$$f(x) = Ax + b$$

A more formal explanation:

$$C = \{s \in R^{p^2} \mid \text{mat}(s) \in S^p \text{ and } \text{mat}(s) \preceq 0\}$$

is convex

$$\begin{aligned} f(x) &= x_1 \text{vec}(A_1) + \cdots + x_m \text{vec}(A_m) - \text{vec}(B) \\ &= \begin{bmatrix} \text{vec}(A_1) & \cdots & \text{vec}(A_m) \end{bmatrix} x + (-\text{vec}(B)) \end{aligned}$$

2-13: Affine Function VI

$$f^{-1}(C) = \{x \mid \text{mat}(f(x)) \in S^p \text{ and } \text{mat}(f(x)) \preceq 0\}$$

is convex

- Hyperbolic cone:

$$C = \{(z, t) \mid z^T z \leq t^2, t \geq 0\}$$

is convex (by drawing a figure in 2 or 3 dimensional space)

2-13: Affine Function VII

- We have that

$$f(x) = \begin{bmatrix} P^{1/2}x \\ c^T x \end{bmatrix} = \begin{bmatrix} P^{1/2} \\ c^T \end{bmatrix} x$$

is affine. Then

$$\begin{aligned} f^{-1}(C) &= \{x \mid f(x) \in C\} \\ &= \{x \mid x^T P x \leq (c^T x)^2, c^T x \geq 0\} \end{aligned}$$

is convex

Perspective and linear-fractional function I

- Image convex: if S is convex, check if

$$\{P(x,t) \mid (x,t) \in S\}$$

  convex or not

  Note that S is in the domain of P

Perspective and linear-fractional function II

- Assume

$$(x_1, t_1), (x_2, t_2) \in S$$

We hope

$$\alpha \frac{x_1}{t_1} + (1 - \alpha) \frac{x_2}{t_2} = P(A, B),$$

where

$$(A, B) \in S$$

Perspective and linear-fractional function III

- We have

$$
\begin{aligned}
\alpha\frac{x_1}{t_1} + (1-\alpha)\frac{x_2}{t_2} &= \frac{\alpha t_2 x_1 + (1-\alpha) t_1 x_2}{t_1 t_2} \\
&= \frac{\alpha t_2 x_1 + (1-\alpha) t_1 x_2}{\alpha t_1 t_2 + (1-\alpha) t_1 t_2} \\
&= \frac{\frac{\alpha t_2}{\alpha t_2 + (1-\alpha) t_1} x_1 + \frac{(1-\alpha) t_1}{\alpha t_2 + (1-\alpha) t_1} x_2}{\frac{\alpha t_2}{\alpha t_2 + (1-\alpha) t_1} t_1 + \frac{(1-\alpha) t_1}{\alpha t_2 + (1-\alpha) t_1} t_2}
\end{aligned}
$$

Perspective and linear-fractional function IV

Let

$$\theta = \frac{\alpha t_2}{\alpha t_2 + (1-\alpha) t_1}$$

We have

$$\frac{\theta x_1 + (1-\theta) x_2}{\theta t_1 + (1-\theta) t_2} = \frac{A}{B}$$

Further

$$(A, B) \in S$$

because

$$(x_1, t_1), (x_2, t_2) \in S$$

Perspective and linear-fractional function V

and

$$S \text{ is convex}$$

- Inverse image is convex
- Given C a convex set

$$P^{-1}(C) = \{(x,t) \mid P(x,t) = x/t \in C\}$$

is convex

Perspective and linear-fractional function VI

- Let

$$
\begin{aligned}
(x_1, t_1) &: x_1/t_1 \in C \\
(x_2, t_2) &: x_2/t_2 \in C
\end{aligned}
$$

Do we have

$$\theta(x_1, t_1) + (1-\theta)(x_2, t_2) \in P^{-1}(C)?$$

That is,

$$\frac{\theta x_1 + (1-\theta)x_2}{\theta t_1 + (1-\theta)t_2} \in C?$$

Perspective and linear-fractional function VII

Let

$$\frac{\theta x_1 + (1-\theta)x_2}{\theta t_1 + (1-\theta)t_2} = \alpha \frac{x_1}{t_1} + (1-\alpha)\frac{x_2}{t_2},$$

Earlier we had

$$\theta = \frac{\alpha t_2}{\alpha t_2 + (1-\alpha)t_1}$$

Then

$$(\alpha(t_2 - t_1) + t_1)\theta = \alpha t_2$$

Perspective and linear-fractional function VIII

$$t_1\theta = \alpha t_2 - \alpha t_2\theta + \alpha t_1\theta$$

$$\alpha = \frac{t_1\theta}{t_1\theta + (1-\theta)t_2}$$

2-16: Generalized inequalities I

- K contains no line:

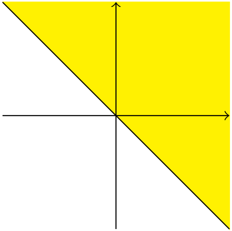
$$\forall x \text{ with } x \in K \text{ and } -x \in K \Rightarrow x = 0$$

- Nonnegative polynomial on $[0, 1]$
- When $n = 2$

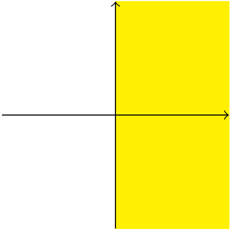
$$x_1 \geq -tx_2, \forall t \in [0, 1]$$

- $t = 1$

2-16: Generalized inequalities II

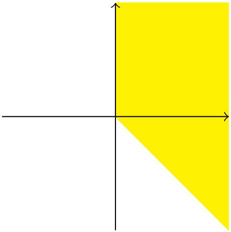


- $t = 0$



2-16: Generalized inequalities III

- $\forall t \in [0, 1]$



- It really becomes a proper cone

2-17: I

- Properties:

$$x \preceq_K y, u \preceq_K v$$

implies that

$$\begin{aligned} x - y &\in K \\ u - v &\in K \end{aligned}$$

- From the definition of a convex cone,

$$(x - y) + (u - v) \in K$$

Then

$$x + u \preceq_K y + v$$

2-18: Minimum and minimal elements I

- The minimum element

$$S \subseteq x_1 + K$$

- A minimal element

$$(x_2 - K) \cap S = \{x_2\}$$

2-19: Separating hyperplane theorem I

- We consider a simplified situation and omit part of the proof
- Assume

$$\inf_{u \in C, v \in D} \|u - v\| > 0$$

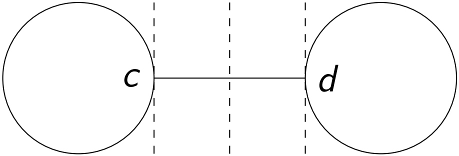
and minimum attained at c, d

- We will show that

$$a \equiv d - c, \quad b \equiv \frac{\|d\|^2 - \|c\|^2}{2}$$

forms a separating hyperplane $a^T x = b$

2-19: Separating hyperplane theorem II



$$a^T x = \Delta_2 \qquad b \qquad \Delta_1$$

$$a = d - c$$

$$\Delta_1 = (d-c)^T d, \quad \Delta_2 = (d-c)^T c$$

$$b = \frac{\Delta_1 + \Delta_2}{2} = \frac{d^T d - c^T c}{2}$$

2-19: Separating hyperplane theorem III

Assume the result is wrong so there is $u \in D$ such that

$$a^T u - b < 0$$

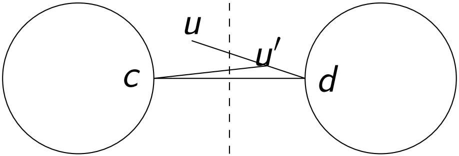
We will derive a point u' in D but closer to c than d. That is,

$$\|u' - c\| < \|d - c\|$$

Then we have a contradiction

- The concept

2-19: Separating hyperplane theorem IV



$$a^T x = b$$

$$a^T u - b = (d-c)^T u - \frac{d^T d - c^T c}{2} < 0$$

implies that

$$(d-c)^T (u-d) + \frac{1}{2}\|d-c\|^2 < 0$$

2-19: Separating hyperplane theorem V

$$
\begin{aligned}
&\frac{d}{dt}\|d + t(u-d) - c\|^2\Big|_{t=0} \\
=&2(d + t(u-d) - c)^T(u-d)\Big|_{t=0} \\
=&2(d-c)^T(u-d) < 0
\end{aligned}
$$

There exists a small $t \in (0,1)$ such that

$$\|d + t(u-d) - c\| < \|d - c\|$$

However,

$$d + t(u-d) \in D,$$

so there is a contradiction

2-20: Supporting hyperplane theorem I

Case 1: C has an interior region

- Consider 2 sets:

$$\text{interior of } C \text{ versus } \{x_0\},$$

  where x_0 is any boundary point
- If C is convex, then interior of C is also convex
- Then both sets are convex
- We can apply results in slide 2-19 so that there exists a such that

$$a^T x \leq a^T x_0, \forall x \in \text{interior of } C$$

2-20: Supporting hyperplane theorem II

- Then for all boundary point x we also have

$$a^T x \leq a^T x_0$$

  because any boundary point is the limit of interior points

Case 2: C has no interior region

- In this situation, C is like a line in R^3 (so no interior). Then of course it has a supporting hyperplane

3-4: Examples on R^n and $R^{m \times n}$ I

$$A, X \in R^{m \times n}$$

$$\text{tr}(A^T X) = \sum_j (A^T X)_{jj}$$

$$= \sum_j \sum_i A^T_{ji} X_{ij} = \sum_j \sum_i A_{ij} X_{ij}$$

3-7: First-order Condition I

- An open set: for any x, there is a ball covering x such that this ball is in the set
- Global underestimator:

$$\frac{z - f(x)}{y - x} = f'(x)$$

$$z = f(x) + f'(x)(y - x)$$

3-7: First-order Condition II

- $\Rightarrow$: Because domain f is convex,

$$\text{for all } 0 < t \leq 1, x + t(y - x) \in \text{ domain } f$$

$$f(x + t(y - x)) \leq (1 - t)f(x) + tf(y)$$

$$f(y) \geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t}$$

when $t \to 0$,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

- $\Leftarrow$:

3-7: First-order Condition III

For any $0 \leq \theta \leq 1$,

$$z = \theta x + (1-\theta) y$$

$$\begin{aligned} f(x) \geq & f(z) + \nabla f(z)^T (x - z) \\ = & f(z) + \nabla f(z)^T (1-\theta)(x-y) \end{aligned}$$

$$\begin{aligned} f(y) \geq & f(z) + \nabla f(z)^T (y - z) \\ = & f(z) + \nabla f(z)^T \theta (y-x) \end{aligned}$$

$$\theta f(x) + (1-\theta) f(y) \geq f(z)$$

3-7: First-order Condition IV

- First-order condition for strictly convex function:

$$f \text{ is strictly convex if and only if}$$
$$f(y) > f(x) + \nabla f(x)^T (y - x)$$

- $\Leftarrow$: it's easy by directly modifying $\geq$ to $>$

$$\begin{aligned} f(x) >& f(z) + \nabla f(z)^T (x - z) \\ =& f(z) + \nabla f(z)^T (1 - \theta)(x - y) \end{aligned}$$

$$f(y) > f(z) + \nabla f(z)^T (y - z) = f(z) + \nabla f(z)^T \theta (y - x)$$

3-7: First-order Condition V

- $\Rightarrow$: Assume the result is wrong. From the 1st-order condition of a convex function, $\exists x, y$ such that $x \neq y$ and

$$\nabla f(x)^T(y-x) = f(y) - f(x) \tag{1}$$

For this (x, y), from the strict convexity

$$\begin{aligned} f(x+t(y-x)) - f(x) &< tf(y) - tf(x) \\ &= \nabla f(x)^T t(y-x), \forall t \in (0,1) \end{aligned}$$

3-7: First-order Condition VI

Therefore,

$$f(x+t(y-x)) < f(x)+\nabla f(x)^T t(y-x), \forall t \in (0,1)$$

However, this contradicts the first-order condition:

$$f(x+t(y-x)) \geq f(x)+\nabla f(x)^T t(y-x), \forall t \in (0,1)$$

This proof was given by a student of this course before

3-8: Second-order condition I

- Proof of the 2nd-order condition:
  We consider only the simpler condition of $n = 1$
- $\Rightarrow$

$$f(x+t) \geq f(x) + f'(x)t$$

$$\lim_{t \to 0} 2\frac{f(x+t) - f(x) - f'(x)t}{t^2}$$

$$= \lim_{t \to 0} \frac{2(f'(x+t) - f'(x))}{2t} = f''(x) \geq 0$$

3-8: Second-order condition II

- "$\Leftarrow$"

$$
\begin{aligned}
f(x+t) &= f(x) + f'(x)t + \frac{1}{2}f''(\bar{x})t^2 \\
&\geq f(x) + f'(x)t
\end{aligned}
$$

  by 1st-order condition
- The extension to general n is straightforward
- If $\nabla^2 f(x) \succ 0$, then f is strictly convex

3-8: Second-order condition III

Using 1st-order condition for strictly convex function:

$$
\begin{aligned}
& f(y) \\
=& f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(\bar{x})(y - x) \\
>& f(x) + \nabla f(x)^T (y - x)
\end{aligned}
$$

Note that in our proof of 1st-order condition for strictly convex functions we used 2nd-order condition of convex function, so no contradiction here

3-8: Second-order condition IV

- It's possible that f is strictly convex but

$$\nabla^2 f(x) \not\succ 0$$

- Example:

$$f(x) = x^4$$

  Details omitted

3-9: Examples I

- Quadratic-over-linear

$$\frac{\partial f}{\partial x} = \frac{2x}{y}, \frac{\partial f}{\partial y} = -\frac{x^2}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{2}{y}, \frac{\partial^2 f}{\partial x \partial y} = -\frac{2x}{y^2}, \frac{\partial^2 f}{\partial y \partial y} = \frac{2x^3}{y^3},$$

$$\frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y & -x \end{bmatrix}$$

$$= \frac{2}{y^3}\begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} = 2\begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ -\frac{x}{y^2} & \frac{x^2}{y^3} \end{bmatrix}$$

3-10 I

$$f(x) = \log \sum_k \exp x_k$$

$$\nabla f(x) = \begin{bmatrix} \frac{e^{x_1}}{\sum_k e^{x_k}} \\ \vdots \\ \frac{e^{x_n}}{\sum_k e^{x_k}} \end{bmatrix}$$

$$\nabla^2_{ii} f = \frac{(\sum_k e^{x_k}) e^{x_i} - e^{x_i} e^{x_i}}{(\sum_k e^{x_k})^2}, \nabla^2_{ij} f = \frac{-e^{x_i} e^{x_j}}{(\sum_k e^{x_k})^2}, i \neq j$$

Note that if

$$z_k = \exp x_k$$

3-10 II

then

$$(zz^T)_{ij} = z_i(z^T)_j = z_i z_j$$

Cauchy-Schwarz inequality

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

$$a_k = v_k \sqrt{z_k}, b_k = \sqrt{z_k}$$

Note that

$$z_k > 0$$

3-12: Jensen's inequality I

- General form

$$f(\int p(z)zdz) \leq \int p(z)f(z)dz$$

- Discrete situation

$$f(\sum p_i z_i) \leq \sum p_i f(z_i), \sum p_i = 1$$

3-12: Jensen's inequality II

- Proof:

$$
\begin{aligned}
& f(p_1 z_1 + p_2 z_2 + p_3 z_3) \\
\leq & (1-p_3)\left(f\left(\frac{p_1 z_1 + p_2 z_2}{1-p_3}\right)\right) + p_3 f(z_3) \\
\leq & (1-p_3)\left(\frac{p_1}{1-p_3} f(z_1) + \frac{p_2}{1-p_3} f(z_2)\right) + p_3 f(z_3) \\
= & p_1 f(z_1) + p_2 f(z_2) + p_3 f(z_3)
\end{aligned}
$$

- Note that

$$
\frac{p_1}{1-p_3} + \frac{p_2}{1-p_3} = \frac{1-p_3}{1-p_3} = 1
$$

Positive weighted sum & composition with affine function I

- Composition with affine function:

  We know

$$f(x) \text{ is convex}$$

  Is

$$g(x) = f(Ax + b)$$

Positive weighted sum & composition with affine function II

convex?

$$
\begin{aligned}
& g((1-\alpha)x_1+\alpha x_2) \\
=& f(A((1-\alpha)x_1+\alpha x_2)+b) \\
=& f((1-\alpha)(Ax_1+b)+\alpha(Ax_2+b)) \\
\leq& (1-\alpha)f(Ax_1+b)+\alpha f(Ax_2+b) \\
=& (1-\alpha)g(x_1)+\alpha g(x_2)
\end{aligned}
$$

3-15: Pointwise maximum I

- Proof of the convexity

$$
\begin{aligned}
& f((1-\alpha)x_1 + \alpha x_2) \\
=& \max(f_1((1-\alpha)x_1 + \alpha x_2), \ldots, f_m((1-\alpha)x_1 + \alpha x_2)) \\
\le& \max((1-\alpha)f_1(x_1) + \alpha f_1(x_2), \ldots, \\
& \quad (1-\alpha)f_m(x_1) + \alpha f_m(x_2)) \\
\le& (1-\alpha)\max(f_1(x_1), \ldots, f_m(x_1)) + \\
& \quad \alpha \max(f_1(x_2), \ldots, f_m(x_2)) \\
\le& (1-\alpha)f(x_1) + \alpha f(x_2)
\end{aligned}
$$

3-15: Pointwise maximum II

- For
  $$f(x) = x_{[1]} + \cdots + x_{[r]}$$
  consider all
  $$\binom{n}{r}$$
  combinations

3-16: Pointwise supremum I

- The proof is similar to pointwise maximum
- Support function of a set C:

  When y is fixed,

  $$f(x, y) = y^T x$$

  is linear (convex) in x
- Maximum eigenvales of symmetric matrix

  $$f(X, y) = y^T X y$$

  is a linear function of X when y is fixed

3-19: Minimization I

- Proof:

  Let $\epsilon > 0$. $\exists y_1, y_2 \in C$ such that

$$f(x_1, y_1) \le g(x_1) + \epsilon$$

$$f(x_2, y_2) \le g(x_2) + \epsilon$$

$$\begin{aligned}
g(\theta x_1 + (1-\theta)x_2) &= \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y) \\
&\le f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\
&\le \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2) \\
&\le \theta g(x_1) + (1-\theta) g(x_2) + \epsilon
\end{aligned}$$

3-19: Minimization II

- Note that the first inequality use the peroperty that C is convex to have

$$\theta y_1 + (1-\theta) y_2 \in C$$

- Because the above inequality holds for all $\epsilon > 0$,

$$g(\theta x_1 + (1-\theta)x_2) \le \theta g(x_1) + (1-\theta) g(x_2)$$

3-19: Minimization III

- first example:
  The goal is to prove

$$A - BC^{-1}B^T \succeq 0$$

Instead of a direct proof, here we use the property in this slide. First we have that $f(x, y)$ is convex in (x, y) because

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$$

Consider

$$\min_{y} f(x, y)$$

3-19: Minimization IV

Because

$$C \succ 0,$$

the minimum occurs at

$$2Cy + 2B^T x = 0$$

$$y = -C^{-1}B^T x$$

Then

$$\begin{aligned} g(x) &= x^T Ax - 2x^T BC^{-1}Bx + x^T BC^{-1}CC^{-1}B^T x \\ &= x^T (A - BC^{-1}B^T)x \end{aligned}$$

3-19: Minimization V

is convex. The second-order condition implies that

$$A - BC^{-1}B^T \succeq 0$$

3-21: the conjugate function I

- This function is useful later
- When y is fixed, maximum happens at

$$y = f'(x) \tag{2}$$

by taking the derivative on x

3-21: the conjugate function II

- Explanation of the figure: when y is fixed

$$z = xy$$

is a straight line passing through the origin, where y is the slope of the line. Check under which x,

$$yx \text{ and } f(x)$$

have the largest distance

- From the figure, the largest distance happens when (2) holds

3-21: the conjugate function III

- About the point

$$(0, -f^*(y))$$

The tangent line is

$$\frac{z - f(x_0)}{x - x_0} = f'(x_0)$$

where x_0 is the point satisfying

$$y = f'(x_0)$$

When $x = 0$,

$$z = -x_0 f'(x_0) + f(x_0) = -x_0 y + f(x_0) = -f^*(y)$$

3-21: the conjugate function IV

- f^* is convex: Given x,

$$y^T x - f(x)$$

  is linear (convex) in y. Then we apply the property of pointwise supremum

3-22: examples I

- negative logarithm

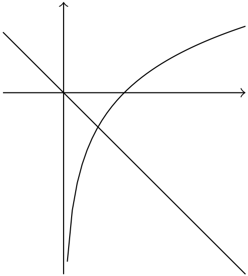
$$f(x) = -\log x$$

$$\frac{\partial}{\partial x}(xy + \log x) = y + \frac{1}{x} = 0$$

If $y < 0$, the picture of

$$xy + \log x$$

3-22: examples II



Then

$$xy + \log x = -1 - \log(-y)$$

3-22: examples III

- strictly convex quadratic

$$Qx = y, x = Q^{-1}y$$

$$\begin{aligned}
&y^T x - \frac{1}{2}x^T Qx \\
=&y^T Q^{-1}y - \frac{1}{2}y^T Q^{-1}QQ^{-1}y \\
=&\frac{1}{2}y^T Q^{-1}y
\end{aligned}$$

3-23: quasiconvex functions I

- Figure on slide:

$$S_\alpha = [a, b], S_\beta = (-\infty, c]$$

  Both are convex
- The figure is an example showing that quasi convex may not be convex

3-26: properties of quasiconvex functions I

- Modified Jensen inequality:
  f quasiconvex if and only if

  $$f(\theta x+(1-\theta)y) \leq \max\{f(x), f(y)\}, \forall x, y, \theta \in [0,1].$$

- $\Rightarrow$ Let

  $$\Delta = \max\{f(x), f(y)\}$$

  S_Δ is convex

  $$x \in S_\Delta, y \in S_\Delta$$

  $$\theta x + (1-\theta)y \in S_\Delta$$

3-26: properties of quasiconvex functions II

$$f(\theta x + (1-\theta)y) \le \Delta$$

and the result is obtained

- $\Leftarrow$ If results are wrong, there exists α such that S_α is not convex.

$$\exists x, y, \theta \text{ with } x, y \in S_\alpha, \theta \in [0,1] \text{ such that}$$

$$\theta x + (1-\theta)y \notin S_\alpha$$

3-26: properties of quasiconvex functions III

Then

$$f(\theta x + (1-\theta)y) > \alpha \geq \max\{f(x), f(y)\}$$

This violates the assumption

- First-order condition (this is exercise 3.43):

3-26: properties of quasiconvex functions IV

"$\Rightarrow$"

$$f((1-t)x+ty) \le \max(f(x), f(y)) = f(x)$$

$$\frac{f(x+t(y-x))-f(x)}{t} \le 0$$

$$\lim_{t\to 0} \frac{f(x+t(y-x))-f(x)}{t} = \nabla f(x)^T(y-x) \le 0$$

3-26: properties of quasiconvex functions V

- $\Leftarrow$: If results are wrong, there exists α such that S_α is not convex.

$$\exists x, y, \theta \text{ with } x, y \in S_\alpha, \theta \in [0,1] \text{ such that}$$

$$\theta x + (1-\theta)y \notin S_\alpha$$

Then

$$f(\theta x + (1-\theta)y) > \alpha \geq \max\{f(x), f(y)\} \tag{3}$$

3-26: properties of quasiconvex functions VI

Because f is differentiable, it is continuous. Without loss of generality, we have

$$f(z) \geq f(x), f(y), \forall z \text{ between } x \text{ and } y$$

$$z = x + \theta(y - x), \theta \in (0, 1)$$

$$\nabla f(z)^T(-\theta(y - x)) \leq 0$$

$$\nabla f(z)^T(y - x - \theta(y - x)) \leq 0$$

Then

$$\nabla f(z)^T(y - x) = 0, \forall \theta \in (0, 1)$$

3-26: properties of quasiconvex functions VII

$$
\begin{aligned}
& f(x+\theta(y-x)) \\
=& f(x)+\nabla f(t)^T \theta(y-x) \\
=& f(x), \forall \theta \in [0,1)
\end{aligned}
$$

This contradicts (3).

3-27: Log-concave and log-convex functions I

- Powers:

$$\log(x^a) = a \log x$$

$$\log x \text{ is concave}$$

- Probability densities:

$$\log f(x) = -\frac{1}{2}(x - \bar{x})^T \Sigma^{-1}(x - \bar{x}) + \text{ constant}$$

Σ^{-1} is positive definite. Thus $\log f(x)$ is concave

3-27: Log-concave and log-convex functions II

- Cumulative Gaussian distribution

$$\log \Phi(x) = \log \int_{-\infty}^{x} e^{-u^2/2} du$$

$$\frac{d}{dx} \log \Phi(x) = \frac{e^{-x^2/2}}{\int_{-\infty}^{x} e^{-u^2/2} du}$$

$$\frac{d^2}{d^2x} \log \Phi(x)$$

$$= \frac{(\int_{-\infty}^{x} e^{-u^2/2} du) e^{-x^2/2}(-x) - e^{-x^2/2} e^{-x^2/2}}{(\int_{-\infty}^{x} e^{-u^2/2} du)^2}$$

3-27: Log-concave and log-convex functions III

Need to prove that

$$\left(\int_{-\infty}^{x} e^{-u^2/2} du\right) x + e^{-x^2/2} > 0$$

- Because

$$x \geq u \text{ for all } u \in (-\infty, x],$$

3-27: Log-concave and log-convex functions IV

we have

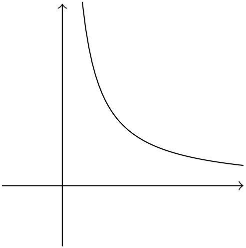
$$
\begin{aligned}
&\left(\int_{-\infty}^{x} e^{-u^2/2} du\right) x + e^{-x^2/2} \\
=& \int_{-\infty}^{x} x e^{-u^2/2} du + e^{-x^2/2} \\
\geq& \int_{-\infty}^{x} u e^{-u^2/2} du + e^{-x^2/2} \\
=& - e^{-u^2/2} |_{-\infty}^{x} + e^{-x^2/2} \\
=& - e^{-x^2/2} + e^{-x^2/2} = 0
\end{aligned}
$$

3-27: Log-concave and log-convex functions V

This proof was given by a student (and polished by another student) of this course before

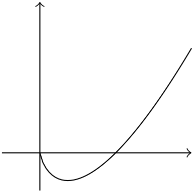
4-3: Optimal and locally optimal points I

- $f_0(x) = 1/x$



- $f_0(x) = x \log x$

4-3: Optimal and locally optimal points II

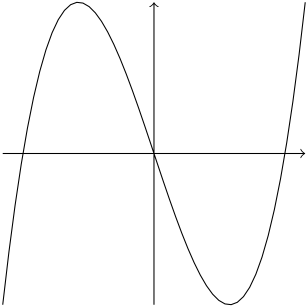


$$f_0'(x) = 1 + \log x = 0$$

$$x = e^{-1} = 1/e$$

- $f_0(x) = x^3 - 3x$

4-3: Optimal and locally optimal points III



$$f_0'(x) = 3x^2 - 3 = 0$$

$$x = \pm 1$$

4-9: Optimality criterion for differentiable f_0 I

- $\Leftarrow$: easy

  From first-order condition

  $$f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y - x)$$

  Together with

  $$\nabla f_0(x)^T(y - x) \geq 0$$

  we have

  $$f_0(y) \geq f_0(x), \text{ for all feasible } y$$

4-9: Optimality criterion for differentiable f_0 II

- $\Rightarrow$ Assume the result is wrong. Then

$$\nabla f_0(x)^T(y-x) < 0$$

Let

$$z(t) = ty + (1-t)x$$

$$\frac{d}{dt} f_0(z(t)) = \nabla f_0(z(t))^T(y-x)$$

$$\left.\frac{d}{dt} f_0(z(t))\right|_{t=0} = \nabla f_0(x)^T(y-x) < 0$$

4-9: Optimality criterion for differentiable f_0 III

There exists t such that

$$f_0(z(t)) < f_0(x)$$

- Note that

$$z(t)$$

is feasible because

$$f_i(z(t)) \leq tf_i(x) + (1-t)f_i(y) \leq 0$$

and

$$A(tx+(1-t)y) = tAx+(1-t)Ay = tb+(1-b)b = b$$

4-9: Optimality criterion for differentiable f_0 IV

4-10 I

- Unconstrained problem:

  Let

  $$y = x - t\nabla f_0(x)$$

  It is feasible (unconstrained problem). Optimality condition implies

  $$\nabla f_0(x)^T(y - x) = -t\|\nabla f_0(x)\|^2 \geq 0$$

  Thus

  $$\nabla f_0(x) = 0$$

- Equality constrained problem

4-10 II

$\Leftarrow$ Easy. For any feasible y,

$$Ay = b$$

$$\nabla f_0(x)(y-x) = -\nu^T A(y-x) = -\nu^T(b-b) = 0 \geq 0$$

So x is optimal

$\Rightarrow$: more complicated. We only do a rough explanation

4-10 III

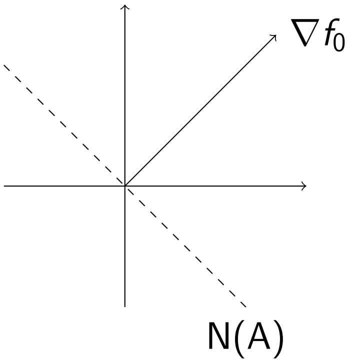
From optimality condition

$$\nabla f_0(x)^T \nu = \nabla f_0(x)^T ((x+\nu) - x) \geq 0, \forall \nu \in N(A)$$

$N(A)$ is a subspace in 2-D. Thus

$$\nu \in N(A) \Rightarrow -\nu \in N(A)$$

4-10 IV



4-10 V

We have

$$\nabla f_0(x)^T \nu = 0, \forall \nu \in N(A)$$

$$\nabla f_0(x) \perp N(A), \nabla f_0(x) \in R(A^T)$$

$$\Rightarrow \exists \nu \text{ such that } \nabla f_0(x) + A^T \nu = 0$$

- Minimization over nonnegative orthant

$\Leftarrow$ Easy

4-10 VI

For any $y \succeq 0$,

$$\nabla_i f_0(x)(y_i - x_i) = \begin{cases} \nabla_i f_0(x) y_i \geq 0 & \text{if } x_i = 0 \\ 0 & \text{if } x_i > 0. \end{cases}$$

Therefore,

$$\nabla f_0(x)^T (y - x) \geq 0$$

and

$$x \text{ is optimal}$$

4-10 VII

$\Rightarrow$ If $x_i = 0$, we claim

$$\nabla_i f_0(x) \geq 0$$

Otherwise,

$$\nabla_i f_0(x) < 0$$

Let

$$y = x \text{ except } y_i \rightarrow \infty$$

$$\nabla f_0(x)^T (y - x) = \nabla_i f_0(x)(y_i - x_i) \rightarrow -\infty$$

This violates the optimality condition

4-10 VIII

If $x_i > 0$, we claim

$$\nabla_i f_0(x) = 0$$

Otherwise, assume

$$\nabla_i f_0(x) > 0$$

Consider

$$y = x \text{ except } y_i = x_i/2 > 0$$

4-10 IX

It is feasible. Then

$$\nabla f_0(x)^T(y-x) = \nabla_i f_0(x)(y_i-x_i) = -\nabla_i f_0(x)x_i/2 < 0$$

violates the optimality condition. The situation for

$$\nabla_i f_0(x) < 0$$

is similar

4-23: examples I

$$\bar{c} \equiv E(C)$$

$$\Sigma \equiv E_C((C - \bar{c})(C - \bar{c}))$$

$$\begin{aligned}
\mathrm{Var}(C^T x) &= E_C((C^T x - \bar{c}^T x)(C^T x - \bar{c}^T x)) \\
&= E_C(x^T (C - \bar{c})(C - \bar{c})^T x) \\
&= x^T \Sigma x
\end{aligned}$$

4-25: second-order cone programming I

- Cone was defined on slide 2-8

$$\{(x, t) \mid \|x\| \leq t\}$$

4-35: generalized inequality constraint I

- $f_i \in R^n \to R^{k_i}$ K_i-convex:

$$f_i(\theta x + (1-\theta)y) \preceq_{K_i} \theta f_i(x) + (1-\theta) f_i(y)$$

- See page 3-31

- LP and equivalent SDP

$$Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_1 \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} & & \\ & \ddots & \\ & & a_{mn} \end{bmatrix}$$

$$- \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_m \end{bmatrix} \preceq 0$$

4-37: LP and SOCP as SDP II

- For SOCP and SDP we will use results in 4-39:

$$\begin{bmatrix} tI_{p\times p} & A_{p\times q} \\ A^T & tI_{q\times q} \end{bmatrix} \succeq 0 \Leftrightarrow A^TA \preceq t^2 I_{q\times q}, t \geq 0$$

- Now

$$\begin{gathered} p = m, q = 1 \\ A = A_i x + b_i, t = c_i^T x + d_i \\ \|A_i x + b_i\|^2 \leq (c_i^T x + d_i)^2, \\ c_i^T x + d_i \geq 0 \end{gathered}$$

- Thus

$$\|A_i x + b_i\| \leq c_i^T x + d_i$$

4-39: matrix norm minimization I

- Following 4-38, we have the following equivalent problem

$$\begin{array}{ll} \min & t \\ \text{subject to} & \|A\|_2 \leq t \end{array}$$

- We then use

$$\begin{aligned} \|A\|_2 \leq t &\Leftrightarrow A^T A \preceq t^2 I, t \geq 0 \\ &\Leftrightarrow \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0 \end{aligned}$$

4-39: matrix norm minimization II

to have the SDP

$$\begin{array}{rl} \min & t \\ \text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

- Next we prove

$$\begin{bmatrix} tI_{p\times p} & A_{p\times q} \\ A^T & tI_{q\times q} \end{bmatrix} \succeq 0 \Leftrightarrow A^T A \preceq t^2 I_{q\times q}, t \geq 0$$

4-39: matrix norm minimization III

- $\Rightarrow$ we immediately have

$$t \geq 0$$

If $t > 0$,

$$\begin{aligned}
&\begin{bmatrix} -v^T A^T & t v^T \end{bmatrix} \begin{bmatrix} tI_{p\times p} & A_{p\times q} \\ A^T & tI_{q\times q} \end{bmatrix} \begin{bmatrix} -Av \\ tv \end{bmatrix} \\
=&\begin{bmatrix} -v^T A^T & t v^T \end{bmatrix} \begin{bmatrix} -tAv + tAv \\ -A^T A v + t^2 v \end{bmatrix} \\
=&t(t^2 v^T v - v^T A^T A v) \geq 0
\end{aligned}$$

$$v^T (t^2 I - A^T A) v \geq 0, \forall v$$

4-39: matrix norm minimization IV

and hence

$$t^2 I - A^T A \succeq 0$$

If $t = 0$

$$\begin{aligned}
&\begin{bmatrix} -v^T A^T & v^T \end{bmatrix} \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} -Av \\ v \end{bmatrix} \\
= &\begin{bmatrix} -v^T A^T & v^T \end{bmatrix} \begin{bmatrix} Av \\ -A^T Av \end{bmatrix} \\
= & -2v^T A^T Av \geq 0, \forall v
\end{aligned}$$

Therefore

$$A^T A \preceq 0$$

4-39: matrix norm minimization V

- $\Leftarrow$ Consider

$$\begin{aligned}
&\begin{bmatrix} u^T & v^T \end{bmatrix} \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
= &\begin{bmatrix} u^T & v^T \end{bmatrix} \begin{bmatrix} tu + Av \\ A^T u + tv \end{bmatrix} \\
= &tu^T u + 2v^T A^T u + tv^T v
\end{aligned}$$

We hope to have

$$tu^T u + 2v^T A^T u + tv^T v \geq 0, \forall (u, v)$$

4-39: matrix norm minimization VI

If $t > 0$

$$\min_u tu^T u + 2v^T A^T u + tv^T v$$

has optimum at

$$u = \frac{-Av}{t}$$

We have

$$\begin{aligned} & tu^T u + 2v^T A^T u + tv^T v \\ = & tv^T v - \frac{v^T A^T A v}{t} \\ = & \frac{1}{t} v^T (t^2 I - A^T A) v \geq 0. \end{aligned}$$

4-39: matrix norm minimization VII

Hence

$$\begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0$$

If $t = 0$

$$A^T A \preceq 0$$

$$v^T A^T A v \leq 0, v^T A^T A v = \|Av\|^2 = 0$$

Thus

$$Av = 0, \forall v$$

$$\begin{bmatrix} u^T & v^T \end{bmatrix} \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} u^T & v^T \end{bmatrix} \begin{bmatrix} 0 \\ A^T u \end{bmatrix} = 0 \geq 0$$

4-39: matrix norm minimization VIII

Thus

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \succeq 0$$

4-40: Vector optimization I

- Though

$$f_0(x) \text{ is a vector}$$

  note that

$$f_i(x) \text{ is still } R^n \rightarrow R^1$$

- K-convex

  See 3-31 though we didn't discuss it earlier

4-41: optimal and pareto optimal points I

- Optimal

$$O \subseteq \{x\} + K$$

- Pareto optimal

$$(x - K) \cap O = \{x\}$$

5-3: Lagrange dual function I

- Note that g is concave no matter if the original problem is convex or not

$$f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x)$$

is convex (linear) in λ, ν for each x

5-3: Lagrange dual function II

Use pointwise supremum on 3-16

$$\sup_{x \in D}(-f_0(x) - \sum \lambda_i f_i(x) - \sum \nu_i h_i(x))$$

is convex. Hence

$$\inf(f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x))$$

is concave. Note that

$$\begin{aligned} -\sup(-\cdots) &= -\text{convex} \\ = \inf(\cdots) &= \text{concave} \end{aligned}$$

5-8: Lagrange dual and conjugate function I

$$
\begin{aligned}
& f_0^*(-A^T\lambda - c^T\nu) \\
=& \sup_x ((-A^T\lambda - c^T\nu)^T x - f_0(x)) \\
=& -\inf_x (f_0(x) + (A^T\lambda + c^T\nu)^T x)
\end{aligned}
$$

5-10: weak and strong duality I

- We don't discuss the SDP problem on this slide because we omitted 5-7

5-11: Slater's constraint qualification I

- We omit the proof because of no time
- "linear inequality do not need to hold with strict inequality": for linear inequalities we DO NOT need constraint qualification
- We will see some explanation later

5-12: inequality from LP I

- If we have only linear constraints, then constraint qualification holds

5-15: geometric interpretation I

- Explanation of $g(\lambda)$: when λ is fixed

  $$\lambda u + t = \Delta$$

  is a line. We lower Δ until it touches the boundary of G

  The Δ value then becomes $g(\lambda)$
- When

  $$u = 0 \Rightarrow t = \Delta$$

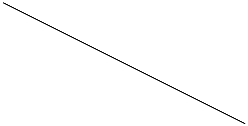
  so we see the point marked as $g(\lambda)$ on t-axis

5-15: geometric interpretation II

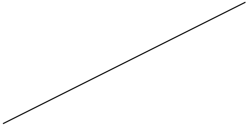
- We have $\lambda \geq 0$, so

$$\lambda u + t = \Delta$$

must be like



rather than



5-15: geometric interpretation III

- Explanation of p^*:
  In G, only points satisfying

$$u \leq 0$$

  are feasible

5-16 I

- We do not discuss a formal proof of

$$\text{Slater condition} \Rightarrow \text{strong duality}$$

  Instead, we explain this result by figures
- Reason of using A: G may not be convex
- Example:

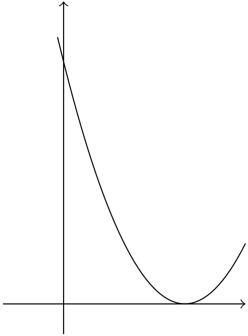
$$\begin{aligned} \min \quad & x^2 \\ \text{subject to} \quad & x + 2 \leq 0 \end{aligned}$$

5-16 II

This is a convex optimization problem

$$G = \{(x+2, x^2) \mid x \in R\}$$

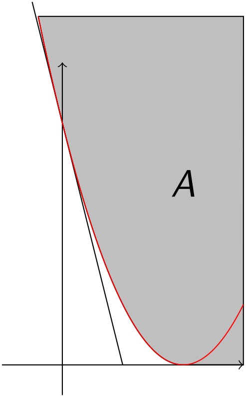
is only a quadratic curve



5-16 III

The curve is not convex

- However, A is convex



5-16 IV

- Primal problem:

$$x = -2$$

  optimal objective value $= 4$

- Dual problem:

$$g(\lambda) = \min_x x^2 + \lambda(x+2)$$

$$x = -\lambda/2$$

$$\max_{\lambda \geq 0} -\frac{\lambda^2}{4} + 2\lambda$$

$$\text{optimal } \lambda = 4$$

5-16 V

$$\text{optimal objective value } = -\frac{16}{4} + 8 = 4$$

- Proving that A is convex

$$(u_1, t_1) \in A, (u_2, t_2) \in A$$

$$\exists x_1, x_2 \text{ such that}$$

$$f_1(x_1) \leq u_1, f_0(x_1) \leq t_1$$

$$f_1(x_2) \leq u_2, f_0(x_2) \leq t_2$$

Consider

$$x = \theta x_1 + (1 - \theta) x_2$$

5-16 VI

We have

$$f_1(x) \le \theta u_1 + (1-\theta)u_2$$

$$f_0(x) \le \theta t_1 + (1-\theta)t_2$$

So

$$\begin{bmatrix} u \\ t \end{bmatrix} = \theta \begin{bmatrix} u_1 \\ t_1 \end{bmatrix} + (1-\theta) \begin{bmatrix} u_2 \\ t_2 \end{bmatrix} \in A$$

- Note that we have

  Slater condition $\Rightarrow$ strong duality

  However, it's possible that Slater condition doesn't hold but strong duality holds

5-16 VII

Example from exercise 5.22:

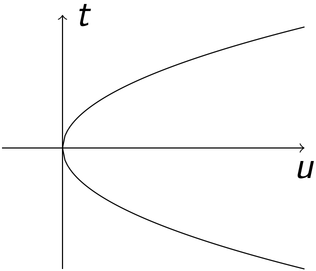
$$\begin{array}{ll} \min & x \\ \text{subject to} & x^2 \leq 0 \end{array}$$

Slater condition doesn't hold because no x satisfies

$$x^2 < 0$$

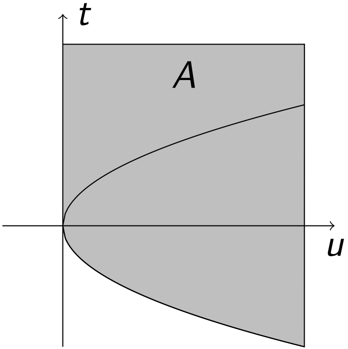
$$G = \{(x^2, x) \mid x \in R\}$$

5-16 VIII



There is only one feasible point $(0, 0)$

5-16 IX



5-16 X

$$g(\lambda) = \min_x x + x^2\lambda$$

$$x = \begin{cases} -1/(2\lambda) & \text{if } \lambda > 0 \\ -\infty & \text{if } \lambda = 0 \end{cases}$$

Dual problem

$$\max_{\lambda \geq 0} -1/(4\lambda)$$

$$\lambda \to \infty, \text{ objective value } \to 0$$

$$d^* = 0, p^* = 0$$

Strong duality holds

5-17: complementary slackness I

- In deriving the inequality we use

$$h_i(x^*) = 0 \text{ and } f_i(x^*) \leq 0$$

- Complementary slackness

  compare the earlier results in 4-10

5-19: KKT conditions for convex problem I

- For the problem on p5-16, neither slater condition nor KKT condition holds

$$1 \neq \lambda 0$$

  Therefore, for convex problems,

$$\text{KKT} \Rightarrow \text{optimality}$$

  but not vice versa.
- Next we explain why for linear constraints we don't need constraint qualification

5-19: KKT conditions for convex problem II

- Consider the situation of inequality constraints only:

$$\begin{array}{rl} \min & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \end{array}$$

- Consider an optimial solution x. We would like to prove that x satisfies KKT condition
- Because x is optimal, from the optimality condition on slide 4-9, for any feasible direction δx,

$$\nabla f_0(x)^T \delta x \geq 0.$$

5-19: KKT conditions for convex problem III

- A feasible δx means

$$f_i(x + \delta x) \leq 0, \forall i$$

- Because

$$f_i(x + \delta x) \approx f_i(x) + \nabla f_i(x)^T \delta x$$

from

$$f_i(x) \leq 0, \forall i$$

we have

$$\nabla f_i(x)^T \delta x \leq 0 \text{ if } f_i(x) = 0.$$

5-19: KKT conditions for convex problem IV

- We claim that

$$\nabla f_0(x) = \sum_{\lambda_i \geq 0, f_i(x)=0} -\lambda_i \nabla f_i(x) \tag{4}$$

- Assume the result is wrong. First let's consider

$$\nabla f_0(x) = \text{linear combination of } \{\nabla f_i(x) \mid f_i(x) = 0\} + \Delta,$$

where

$$\Delta \neq 0 \text{ and } \Delta^T \nabla f_i(x) = 0, \forall i : f_i(x) = 0$$

5-19: KKT conditions for convex problem V

- Then there exists $\alpha < 0$ such that

$$\delta x \equiv \alpha \Delta$$

satisfies

$$\nabla f_i(x)^T \delta x = 0 \text{ if } f_i(x) = 0$$

$$f_i(x + \delta x) \leq 0 \text{ if } f_i(x) < 0$$

and

$$\nabla f_0(x)^T \delta x = \alpha \Delta^T \Delta < 0$$

This contradicts the optimality condition

5-19: KKT conditions for convex problem VI

- By a similar setting we can further prove (4).
- Assume

$$\nabla f_0(x) = \sum_{i:f_i(x)=0} -\lambda_i \nabla f_i(x)$$

and there exists i' such that

$$\lambda_{i'} < 0, \nabla f_{i'}(x) \neq 0, \text{ and } f_{i'}(x) = 0$$

5-19: KKT conditions for convex problem VII

- Let

$$\bar{\lambda} = \arg\min_{\lambda} \|\nabla f_{i'}(x) - \sum_{i:i \neq i', f_i(x)=0} \nabla f_i(x)\lambda_i\|$$

$$\Delta = \|\nabla f_{i'}(x) - \sum_{i:i \neq i', f_i(x)=0} \nabla f_i(x)\bar{\lambda}_i\|$$

  Then

$$\nabla f_i(x)^T \Delta = 0, \forall i \neq i', f_i(x) = 0 \tag{5}$$

5-19: KKT conditions for convex problem VIII

- We have

  $$\nabla f_{i'}(x)^T \Delta \geq 0$$

  If

  $$\nabla f_{i'}(x)^T \Delta = 0$$

  then

  $$-\lambda_{i'} \nabla f_{i'}(x)$$

  can be rearranged to use linear combination of

  $$\{\nabla f_i(x) \mid i \neq i', f_i(x) = 0\}$$

5-19: KKT conditions for convex problem IX

Otherwise

$$\nabla f_{i'}(x)^T \Delta > 0$$

- Let

$$\delta x = \alpha \Delta, \alpha < 0.$$

- From (5),

$$\nabla f_i(x)^T \delta x = 0, \forall i \neq i', f_i(x) = 0$$

$$\nabla f_{i'}(x)^T \delta x = \alpha \nabla f_{i'}(x)^T \Delta < 0.$$

Hence δx is a feasible direction.

5-19: KKT conditions for convex problem X

- However,

$$\nabla f_0(x)^T \delta x = -\alpha \lambda_{i'} \nabla f_{i'}(x)^T \Delta < 0$$

  contradicts the optimality condition
- This proof is not rigourous because of $\approx$
- For linear the proof becomes rigourous

5-25 I

- Explanation of $f_0^*(\nu)$

$$\begin{aligned} & \inf_y (f_0(y) - \nu^T y) \\ = & -\sup_y (\nu^T y - f_0(y)) = -f_0^*(\nu) \end{aligned}$$

where $f_0^*(\nu)$ is the conjugate function

5-26 I

- The original problem

$$g(\lambda, \nu) = \inf_x \|Ax - b\| = \text{constant}$$

- Dual norm:

$$\|\nu\|_* \equiv \sup\{\nu^T y \mid \|y\| \le 1\}$$

If $\|\nu\|_* > 1$,

$$\nu^T y^* > 1, \|y^*\| \le 1$$

5-26 II

$$\begin{aligned}&\inf \|y\| + \nu^T y\\ \le&\|-y^*\| - \nu^T y^* < 0\end{aligned}$$

$$\|-ty^*\| - \nu^T(ty^*) \to -\infty \text{ as } t \to \infty$$

Hence

$$\inf_y \|y\| + \nu^T y = -\infty$$

If $\|\nu\|_* \le 1$, we claim that

$$\inf_y \|y\| + \nu^T y = 0$$

$$y = 0 \Rightarrow \|y\| + \nu^T y = 0$$

5-26 III

If $\exists y$ such that

$$\|y\| + \nu^T y < 0$$

then

$$\|-y\| < -\nu^T y$$

We can scale y so that

$$\sup\{\nu^T y \mid \|y\| \leq 1\} > 1$$

but this causes a contradiction

5-27: implicit constraint I

- The dual function

$$
\begin{aligned}
& c^T x + \nu^T (Ax - b) \\
= & -b^T \nu + x^T (A^T \nu + c)
\end{aligned}
$$

$$
\inf_{-1 \le x_i \le 1} x_i (A^T \nu + c)_i = -|(A^T \nu + c)_i|
$$

5-30: semidefinite program I

- From 5-29 we need that Z is non-negative in the dual cone of S_+^k
- Dual cone of S_+^k is S_+^k (we didn't discuss dual cone so we assume this result)
- Why

$$\mathrm{tr}(Z(\cdots))?$$

We are supposed to do component-wise produt between

$$Z \text{ and } x_1F_1 + \cdots + x_nF_n - G$$

5-30: semidefinite program II

Trace is the component-wise product

$$
\begin{aligned}
& \operatorname{tr}(AB) \\
= & \sum_i (AB)_{ii} \\
= & \sum_i \sum_j A_{ij} B_{ji} = \sum_i \sum_j A_{ij} B_{ij}
\end{aligned}
$$

Note that we take the property that B is symmetric

7-4 I

- Uniform noise

$$p(z) = \begin{cases} \frac{1}{2a} & \text{if } |z| \le a \\ 0 & \text{otherwise} \end{cases}$$

8-10: Dual of maximum margin problem I

Largangian:

$$
\begin{aligned}
&\frac{\|a\|}{2} - \sum_i \lambda_i(a^T x_i + b - 1) + \sum_i \mu_i(a^T y_i + b + 1) \\
=&\frac{\|a\|}{2} + a^T(-\sum_i \lambda_i x_i + \sum_i \mu_i y_i) \\
&\quad + b(-\sum_i \lambda_i + \sum_i \mu_i) + \sum_i \lambda_i + \sum_i \mu_i
\end{aligned}
$$

Because of

$$
b(-\sum_i \lambda_i + \sum_i \mu_i)
$$

8-10: Dual of maximum margin problem II

we have

$$\inf_{a,b} L$$

$$= \begin{cases} \inf_a \frac{\|a\|}{2} - \sum_i \lambda_i a^T x_i + \sum_i \mu_i a^T y_i & \text{if } \sum_i \lambda_i = \sum_i \mu_i \\ -\infty & \text{if } \sum_i \lambda_i \neq \sum_i \mu_i \end{cases}$$

For

$$\inf_a \frac{\|a\|}{2} - \sum_i \lambda_i a^T x_i + \sum_i \mu_i a^T y_i$$

8-10: Dual of maximum margin problem III

we can denote it as

$$\inf_{a} \frac{\|a\|}{2} + v^T a$$

where v is a vector. We cannot do derivative because $\|a\|$ is not differentiable. Formal solution:

8-10: Dual of maximum margin problem IV

- Case 1: If $\|v\| \leq 1/2$:

$$a^T v \geq -\|a\|\|v\| \geq -\frac{\|a\|}{2}$$

so

$$\inf_a \frac{\|a\|}{2} + v^T a \geq 0.$$

However,

$$a = 0 \rightarrow \frac{\|a\|}{2} + v^T a = 0$$

8-10: Dual of maximum margin problem V

Therefore

$$\inf_a \frac{\|a\|}{2} + v^T a = 0.$$

- If $\|v\| > 1/2$, let

$$a = \frac{-tv}{\|v\|}$$

$$\begin{aligned}
&\frac{\|a\|}{2} + v^T a \\
=&\frac{t}{2} - t\|v\| \\
=&t(\frac{1}{2} - \|v\|) \to \infty \text{ if } t \to -\infty
\end{aligned}$$

8-10: Dual of maximum margin problem VI

Thus

$$\inf_a \frac{\|a\|}{2} + v^T a = -\infty$$

8-14

$$\theta = \begin{bmatrix} \text{vec}(P) \\ q \\ r \end{bmatrix}, F(z) = \begin{bmatrix} \vdots \\ z_i z_j \\ \vdots \\ z_i \\ \vdots \\ 1 \end{bmatrix}$$

10-3: initial point and sublevel set I

- The condition that S is closed if

$$f(x) \to \infty \text{ as } x \to \text{ boundary of domain } f$$

Proof: if not, consider

$$\{x_i\} \subset S$$

such that

$$x_i \to \text{ boundary}$$

Then

$$f(x_i) \to \infty > f(x^0)$$

10-3: initial point and sublevel set II

and

$$S \text{ is not closed}$$

- Example

$$f(x) = \log\left(\sum_{i=1}^{m} \exp(a_i^T x + b_i)\right)$$

$$\text{domain } = R^n$$

10-3: initial point and sublevel set III

- Example

$$f(x) = -\sum_i \log(b_i - a_i^T x)$$

$$\text{domain } \neq R^n$$

We use the condition that

$$f(x) \to \infty \text{ as } x \to \text{ boundary of domain } f$$

10-4: strong convexity and implications I

- S is bounded. Otherwise, there exists a set

$$\{y_i \mid y_i = x + \Delta_i\} \subset S$$

satisfying

$$\lim_{i \to \infty} |\Delta_i| = \infty$$

Then

$$f(y_i) \geq f(x) + \nabla f(x)^T \Delta_i + \frac{m}{2} \|\Delta_i\|^2 \to \infty$$

This contradicts

$$f(y) \leq f(x^0)$$

10-4: strong convexity and implications II

- Proof of

$$p^* > -\infty$$

and

$$f(x) - p^* \leq \frac{1}{2m} \|\nabla f(x)\|^2$$

From

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} \|x - y\|^2$$

Minimize the right-hand side with respect to y

$$\nabla f(x) + m(y - x) = 0$$

10-4: strong convexity and implications III

$$\tilde{y} = x - \frac{\nabla f(x)}{m}$$

$$f(y) \geq f(x) + \nabla f(x)^T(\tilde{y} - x) + \frac{m}{2}\|\tilde{y} - x\|^2$$
$$= f(x) - \frac{1}{2m}\|\nabla f(x)\|^2, \forall y$$

Then

$$p^* \geq f(x) - \frac{1}{2m}\|\nabla f(x)\|^2$$

and

$$f(x) - p^* \leq \frac{1}{2m}\|\nabla f(x)\|^2$$

10-5: descent methods I

- If

  $$f(x + t\Delta x) < f(x)$$

  then

  $$\nabla f(x)^T \Delta x < 0$$

  Proof: From the first-order condition of a convex function

  $$f(x + t\Delta x) \geq f(x) + t\nabla f(x)^T \Delta x$$

  Then

  $$t\nabla f(x)^T \Delta x \leq f(x + t\Delta x) - f(x) < 0$$

10-6: line search types I

- Why

$$\alpha \in (0, \frac{1}{2})?$$

  The use of $1/2$ is for convergence though we won't discuss details

- Finite termination of backtracking line search. We argue that $\exists t^* > 0$ such that

$$f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x, \forall t \in (0, t^*)$$

  Otherwise,

$$\exists \{t_k\} \to 0$$

10-6: line search types II

such that

$$f(x + t_k \Delta x) \geq f(x) + \alpha t_k \nabla f(x)^T \Delta x, \forall k$$

$$\begin{aligned} &\lim_{t_k \to 0} \frac{f(x + t_k \Delta x) - f(x)}{t_k} \\ =&\nabla f(x)^T \Delta x \geq \alpha \nabla f(x)^T \Delta x \end{aligned}$$

However,

$$\nabla f(x)^T \Delta x < 0 \text{ and } \alpha > 0$$

cause a contradiction

10-6: line search types III

- Geometric interpretation: the tangent line passes through $(0, f(x))$, so the equation is

$$\frac{y - f(x)}{t - 0} = \nabla f(x)^T \Delta x$$

Because

$$\nabla f(x)^T \Delta x < 0,$$

we see that the line of

$$f(x) + \alpha t \nabla f(x)^T \Delta x$$

10-6: line search types IV

is above that of

$$f(x) + t\nabla f(x)^T \Delta x$$

10-7 I

- Linear convergence. We consider exact line search; proof for backtracking line search is more complicated
- S closed and bounded

$$\nabla^2 f(x) \preceq MI, \forall x \in S$$

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{M}{2} \|y - x\|^2$$

Solve

$$\min_t f(x) - t \nabla f(x)^T \nabla f(x) + \frac{t^2 M}{2} \nabla f(x)^T \nabla f(x)$$

10-7 II

$$t = \frac{1}{M}$$

$$f(x_{\text{next}}) \leq f(x - \frac{1}{M}\nabla f(x)) \leq f(x) - \frac{1}{2M}\nabla f(x)^T \nabla f(x)$$

The first inequality is from the fact that we use exact line search

$$f(x_{\text{next}}) - p^* \leq f(x) - p^* - \frac{1}{2M}\nabla f(x)^T \nabla f(x)$$

From slide 10-4,

$$-\|\nabla f(x)\|^2 \leq -2m(f(x) - p^*)$$

10-7 III

Hence

$$f(x_{\text{next}}) - p^* \leq (1 - \frac{m}{M})(f(x) - p^*)$$

10-8 I

- Assume

$$x_1^k = \gamma(\frac{\gamma-1}{\gamma+1})^k, x_2^k = (-\frac{\gamma-1}{\gamma+1})^k,$$

$$\nabla f(x_1, x_2) = \begin{bmatrix} x_1 \\ \gamma x_2 \end{bmatrix}$$

$$\min_t \frac{1}{2}((x_1 - tx_1)^2 + \gamma(x_2 - t\gamma x_2)^2)$$

$$\min_t \frac{1}{2}(x_1^2(1-t)^2 + \gamma x_2^2(1-t\gamma)^2)$$

10-8 II

$$-x_1^2(1-t)+\gamma x_2^2(1-t\gamma)(-\gamma)=0$$

$$-x_1^2+tx_1^2-\gamma^2x_2^2+\gamma^3tx_2^2=0$$

$$t(x_1^2+\gamma^3x_2^2)=x_1^2+\gamma^2x_2^2$$

$$t=\frac{x_1^2+\gamma^2x_2^2}{x_1^2+\gamma^3x_2^2}=\frac{\gamma^2(\frac{\gamma-1}{\gamma+1})^{2k}+\gamma^2(\frac{\gamma-1}{\gamma+1})^{2k}}{\gamma^2(\frac{\gamma-1}{\gamma+1})^{2k}+\gamma^3(\frac{\gamma-1}{\gamma+1})^{2k}}$$

$$=\frac{2\gamma^2}{\gamma^2+\gamma^3}=\frac{2}{1+\gamma}$$

$$x^{k+1}=x^k-t\nabla f(x^k)=\begin{bmatrix}x_1^k(1-t)\\x_2^k(1-\gamma t)\end{bmatrix}$$

10-8 III

$$x_1^{k+1} = \gamma(\frac{\gamma-1}{\gamma+1})^k(\frac{\gamma-1}{1+\gamma}) = \gamma(\frac{\gamma-1}{\gamma+1})^{k+1}$$

$$\begin{aligned} x_2^{k+1} &= (-\frac{\gamma-1}{\gamma+1})^k(1-\frac{2\gamma}{1+\gamma}) \\ &= (-\frac{\gamma-1}{\gamma+1})^k(\frac{1-\gamma}{1+\gamma}) = (-\frac{\gamma-1}{\gamma+1})^{k+1} \end{aligned}$$

- Why gradient is orghogonal to the tangent line of the contour curve?

10-8 IV

Assume $f(g(t))$ is the countour with

$$g(0) = x$$

Then

$$0 = f(g(t)) - f(g(0))$$

$$\begin{aligned} 0 &= \lim_{t \to 0} \frac{f(g(t)) - f(g(0))}{t} \\ &= \lim_{t \to 0} \nabla f(g(t))^T \nabla g(t) \\ &= \nabla f(x)^T \nabla g(0) \end{aligned}$$

10-8 V

where

$$x + t\nabla g(0)$$

is the tangent line

10-10 I

- linear convergence: from slide 10-7

$$f(x^k) - p^* \le c^k(f(x^0) - p^*)$$

$$\log(c^k(f(x^0) - p^*)) = k \log c + \log(f(x^0) - p^*)$$

is a straight line. Note that now k is the x-axis

10-11: steepest descent method I

- (unnormalized) steepest descent direction:

$$\Delta x_{\text{sd}} = \|\nabla f(x)\|_* \Delta x_{nsd}$$

  Here $\|\cdot\|_*$ is the dual norm
- We didn't discuss much about dual norm, but we can still explain some examples on 10-12

10-12 I

- Euclidean: Δx_{nsd} is by solving

$$\begin{array}{ll} \min & \nabla f^T v \\ \text{subject to} & \|v\| = 1 \end{array}$$

$$\nabla f^T v = \|\nabla f\| \|v\| \cos\theta = -\|\nabla f\| \text{ when } \cos\theta = \pi$$

$$\Delta x_{\text{nsd}} = \frac{-\nabla f(x)}{\|\nabla f(x)\|}$$

$$\|\nabla f(x)\|_* = \|\nabla f(x)\|$$

$$\|\nabla f(x)\|_* \Delta x_{\text{nsd}} = \|\nabla f(x)\|_* \frac{-\nabla f(x)}{\|\nabla f(x)\|} = -\nabla f(x)$$

10-12 II

- Quadratic norm: Δx_{nsd} is by solving

$$\begin{array}{rl} \min & \nabla f^T v \\ \text{subject to} & v^T P v = 1 \end{array}$$

Now

$$\|v\|_P = \sqrt{v^T P v},$$

where P is symmetric positive definite

10-12 III

- Let

$$w = P^{1/2} v$$

The optimization problem becomes

$$\begin{aligned} \min_{w} \quad & \nabla f^T P^{-1/2} w \\ \text{subject to} \quad & \|w\| = 1 \end{aligned}$$

$$\begin{aligned} \text{optimal } w &= \frac{-P^{-1/2}\nabla f}{\|P^{-1/2}\nabla f\|} \\ &= \frac{-P^{-1/2}\nabla f}{\sqrt{\nabla f^T P^{-1} \nabla f}} \end{aligned}$$

10-12 IV

$$\text{optimal } v = \frac{-P^{-1}\nabla f}{\sqrt{\nabla f^T P^{-1} \nabla f}}$$

- Dual norm

$$\|z\|_* = \|P^{-1/2} z\|$$

Therefore

$$\begin{aligned} \Delta x_{\text{sd}} &= \sqrt{\nabla f^T P^{-1} \nabla f} \frac{-P^{-1}\nabla f}{\sqrt{\nabla f^T P^{-1} \nabla f}} \\ &= -P^{-1}\nabla f \end{aligned}$$

10-12 V

- Explanation of the figure:

$$-\nabla f(x)^T \Delta x_{\text{nsd}} = \|-\nabla f(x)\| \|\Delta x_{\text{nsd}}\| \cos\theta$$

  $\|-\nabla f(x)\|$ is a constant. From a point Δx_{nsd} on the boundary, the projected point on $-\nabla f(x)$ indicates

$$\|\Delta x_{\text{nsd}}\| \cos\theta$$

  In the figure, we see that the chosen Δx_{nsd} has the largest $\|\Delta x_{\text{nsd}}\| \cos\theta$

- We omit the discusssion of l_1-norm

10-13 I

- The two figures are by using two P matrices
- The left one has faster convergence
- Gradient descent after change of variables

$$\bar{x} = P^{1/2}x, x = P^{-1/2}\bar{x}$$

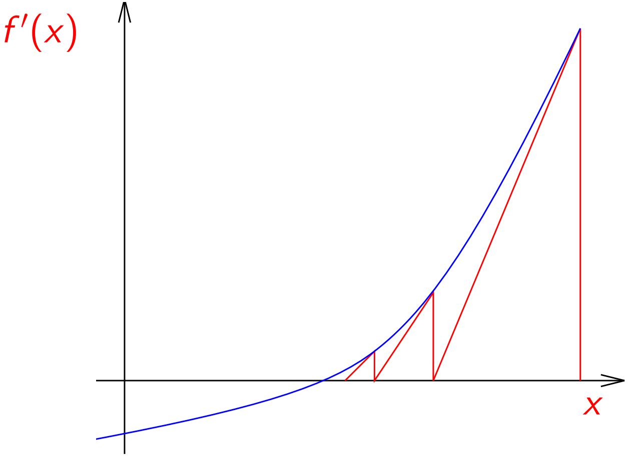
$$\min_x f(x) \Rightarrow \min_{\bar{x}} f(P^{-1/2}\bar{x})$$

$$\bar{x} \leftarrow \bar{x} - \alpha P^{-1/2}\nabla_x f(P^{-1/2}\bar{x})$$

$$P^{1/2}x \leftarrow P^{1/2}x - \alpha P^{-1/2}\nabla_x f(x)$$

$$x \leftarrow x - \alpha P^{-1}\nabla_x f(x)$$

10-14 I



10-14 II

- Solve

$$f'(x) = 0$$

Fining the tangent line at x_k:

$$\frac{y - f'(x_k)}{x - x_k} = f''(x_k)$$

x_k: the current iterate
Let $y = 0$

$$\boxed{x_{k+1} = x_k - f'(x_k)/f''(x_k)}$$

10-16 I

$$\hat{f}(y) = f(x) + \nabla f(x)^T (y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(x)(y-x)$$

$$\nabla \hat{f}(y) = 0 = \nabla f(x) + \nabla^2 f(x)(y-x)$$

$$y - x = -\nabla^2 f(x)^{-1} \nabla f(x)$$

$$\inf_y \hat{f}(y) = f(x) - \frac{1}{2} \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$$

$$f(x) - \inf_y \hat{f}(y) = \frac{1}{2}\lambda(x)^2$$

10-16 II

Norm of the Newton step in the quadratic Hessian norm

$$\Delta x_{\text{nt}} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

$$\Delta x_{\text{nt}}^T \nabla^2 f(x) \Delta x_{\text{nt}} = \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x) = \lambda(x)^2$$

Directional derivative in the Newton direction

$$\begin{aligned} &\lim_{t \to 0} \frac{f(x + t\Delta_{\text{nt}}) - f(x)}{t} \\ =&\nabla f(x)^T \Delta x_{\text{nt}} \\ =& -\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x) = -\lambda(x)^2 \end{aligned}$$

10-16 III

Affine invariant

$$\bar{f}(y) \equiv f(Ty) = f(x)$$

Assume T is an invertable square matrix. Then

$$\bar{\lambda}(y) = \lambda(Ty)$$

Proof:

$$\nabla \bar{f}(y) = T^T \nabla f(Ty)$$
$$\nabla^2 \bar{f}(y) = T^T \nabla^2 f(Ty) T$$

10-16 IV

$$\begin{aligned}
\bar{\lambda}(y)^2 &= \nabla \bar{f}(y)^T \nabla^2 \bar{f}(y)^{-1} \nabla \bar{f}(y) \\
&= \nabla f(Ty)^T T T^{-1} \nabla^2 f(Ty)^{-1} T^{-T} T^T \nabla f(Ty) \\
&= \nabla f(Ty)^T \nabla^2 f(Ty)^{-1} \nabla f(Ty) \\
&= \lambda(Ty)^2
\end{aligned}$$

10-17 I

Affine invariant

$$\begin{aligned}\Delta y_{\text{nt}} &= \nabla^2 \bar{f}(y)^{-1} \nabla \bar{f}(y)\\ &= T^{-1} \nabla^2 f(Ty)^{-1} \nabla f(Ty)\\ &= T^{-1} \Delta x_{\text{nt}}\end{aligned}$$

Note that

$$y_k = T^{-1} x_k$$

so

$$y_{k+1} = T^{-1} x_{k+1}$$

10-17 II

But how about line search

$$
\begin{aligned}
& \nabla \bar{f}(y)^T \Delta y_{\text{nt}} \\
= & \nabla f(Ty)^T T T^{-1} \Delta x_{\text{nt}} \\
= & \nabla f(x)^T \Delta x_{\text{nt}}
\end{aligned}
$$

10-19 I

$$\eta \in (0, \frac{m^2}{L})$$

$$\|\nabla f(x_k)\| \leq \eta \leq \frac{m^2}{L}$$

$$\frac{L}{2m^2}\|\nabla f(x_k)\| \leq \frac{1}{2}$$

10-20 I

$$
\begin{aligned}
& f(x_l) - f(x^*) \\
\le & \frac{1}{2m} \|\nabla f(x_l)\|^2 \qquad \text{(from p10-4)} \\
\le & \frac{1}{2m} \frac{4m^4}{L^2} (\frac{1}{2})^{2^{l-k} \cdot 2} \\
= & \frac{2m^3}{L^2} (\frac{1}{2})^{2^{l-k+1}} \le \epsilon
\end{aligned}
$$

Let

$$\epsilon_0 = \frac{2m^3}{L^2}$$

10-20 II

$$
\begin{aligned}
\log_2 \epsilon_0 - 2^{l-k+1} &\le \log_2 \epsilon \\
2^{l-k+1} &\ge \log_2(\epsilon_0/\epsilon) \\
l &\ge k - 1 + \log_2 \log_2(\epsilon_0/\epsilon) \\
k &\le \frac{f(x_0) - p^*}{r}
\end{aligned}
$$

In at most

$$\frac{f(x_0) - p^*}{r} + \log_2 \log_2(\epsilon_0/\epsilon)$$

iterations, we have

$$f(x_l) - f(x^*) \le \epsilon$$

10-29: implementation I

$$
\begin{aligned}
\lambda(x) &= (\nabla f(x) \nabla^2 f(x)^{-1} \nabla f(x))^{1/2} \\
&= (g^T L^{-T} L^{-1} g)^{1/2} = \|L^{-1} g\|_2
\end{aligned}
$$

10-30: example of dense Newton systems with structure I

$$\nabla f(x) = \begin{bmatrix} \psi_1'(x_1) \\ \vdots \\ \psi_n'(x_n) \end{bmatrix} + A^T \nabla \psi_0(Ax+b)$$

$$\nabla^2 f(x) = \begin{bmatrix} \psi_1''(x_1) & & \\ & \ddots & \\ & & \psi_n''(x_n) \end{bmatrix} + A^T \nabla^2 \psi_0^2(Ax+b)A$$

method 2:

$$\Delta x = D^{-1}(-g - A^T L_o w)$$

10-30: example of dense Newton systems with structure II

$$L_0^T A D^{-1}(-g - A^T L_0 w) = w$$
$$(I + L_0^T A D^{-1} A^T L_0) w = -L_0^T A D^{-1} g$$

Cost

$$L_0 : p \times p$$
$$A^T L_0 : n \times p, \text{cost} : O(np)$$
$$(L_0^T A) D^{-1} (A^T L_0) : O(p^2 n)$$

10-30: example of dense Newton systems with structure III

Note that Cholesky factorization of H_0 costs

$$\frac{1}{3}p^3 \leq p^2 n$$

as

$$p \ll n$$