Numerical Methods 2026 — Final Exam

Solutions

Problem 1 (5 pts). Consider the following 4×6 sparse matrix

$$A = \begin{bmatrix} 0 & 0 & -3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & -1 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume we use **0-based indexing** (i.e., row indices and column indices start from 0).

Please represent the matrix A in both

- **Compressed Sparse Row (CSR)** format and
- **Compressed Sparse Column (CSC)** format.

You need to write out the three 1D arrays

`values`, `indices`, and `pointers`

with respect to the corresponding sparse format.

Solution.

- **CSR Format.** The matrix is processed row by row.
  - `val` stores the non-zero values from top to bottom, left to right.
  - `col_ind` stores the column index of each non-zero value.
  - `row_ptr` stores the starting index of each row in the `val` array. Since there are 4 rows, the length of `row_ptr` is $4 + 1 = 5$.

$$\begin{aligned} \texttt{val} &= [-3, 9, 4, 2, -1, 8] \\ \texttt{col\_ind} &= [2, 5, 0, 2, 4, 1] \\ \texttt{row\_ptr} &= [0, 2, 2, 5, 6] \end{aligned}$$

  `row_ptr[2]` is 2 because the 1st row is completely empty, meaning the next non-zero element (for the 2nd row) still starts at index 2.

- **CSC Format.** The matrix is processed column by column.

- `val` stores the non-zero values from left to right, top to bottom.
- `row_ind` stores the row index of each non-zero value.
- `col_ptr` stores the starting index of each column in the `val` array. Since there are 6 columns, the length of `col_ptr` is $6 + 1 = 7$.

$$\begin{aligned} \texttt{val} &= [4, 8, -3, 2, -1, 9] \\ \texttt{row\_ind} &= [2, 3, 0, 2, 2, 0] \\ \texttt{col\_ptr} &= [0, 1, 2, 4, 4, 5, 6] \end{aligned}$$

`col_ptr[4]` is 4 because the 3rd column is completely empty.

Problem 2 (35 pts). Given n points

$$\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n \in \mathbb{R}^m$$

and n scalar values

$$y_1, y_2, \cdots, y_n.$$

Consider the least squares problem where we want to find a $\boldsymbol{w} \in \mathbb{R}^m$ such that

$$\min_{\boldsymbol{w}} f(\boldsymbol{w}) = \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{w}^T\boldsymbol{x}_i - y_i)^2.$$

Please answer the following questions.

(a) (10 pts) We can arrange $\boldsymbol{x}_i$ and y_i to form

$$X = \begin{bmatrix} \boldsymbol{x}_1^T \\ \boldsymbol{x}_2^T \\ \vdots \\ \boldsymbol{x}_n^T \end{bmatrix} \text{ and } \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

such that

$$f(\boldsymbol{w}) = \frac{1}{2}\|X\boldsymbol{w} - \boldsymbol{y}\|_2^2. \tag{1}$$

Now find the gradient $\nabla_{\boldsymbol{w}} f(\boldsymbol{w})$ and the Hessian matrix $\nabla^2_{\boldsymbol{w}} f(\boldsymbol{w})$. Recall that for a differentiable function $f : \mathbb{R}^m \to \mathbb{R}$, the gradient is defined as

$$\nabla_{\boldsymbol{w}} f(\boldsymbol{w}) = \begin{bmatrix} \frac{\partial f}{\partial w_1} \\ \frac{\partial f}{\partial w_2} \\ \cdots \\ \frac{\partial f}{\partial w_m} \end{bmatrix},$$

and the Hessian matrix is defined as

$$\nabla^2_{\boldsymbol{w}} f(\boldsymbol{w}) = \begin{bmatrix} \frac{\partial^2 f}{\partial w_1^2} & \frac{\partial^2 f}{\partial w_1 \partial w_2} & \cdots & \frac{\partial^2 f}{\partial w_1 \partial w_m} \\ \frac{\partial^2 f}{\partial w_2 \partial w_1} & \frac{\partial^2 f}{\partial w_2^2} & \cdots & \frac{\partial^2 f}{\partial w_2 \partial w_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial w_m \partial w_1} & \frac{\partial^2 f}{\partial w_m \partial w_2} & \cdots & \frac{\partial^2 f}{\partial w_m^2} \end{bmatrix}.$$

Show how you derive the gradient and Hessian matrix and express them in terms of X, $\boldsymbol{y}$, and $\boldsymbol{w}$.

(b) (5 pts) Assume that the Hessian matrix is invertible. In the classroom, Prof. Lin showed that (1) has a closed-form solution $\boldsymbol{w}^*$. Reproduce the derivation by using your result in (a) and express the $\boldsymbol{w}^*$ in terms of X and $\boldsymbol{y}$.

(c) (10 pts) During the derivation from (b), we can get a linear system similar to the following form:

$$A\boldsymbol{w} = \boldsymbol{b}, \text{where } A \text{ is not an identity matrix.}$$

Now, we have five data points in $\mathbb{R}^2$,

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and five scalar values

$$y_1 = 0, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = 1, \quad y_5 = 0.$$

Please use the given data points and scalar values to construct the linear system based on your derivation in part (b). Then, solve the resulting linear system using the Conjugate Gradient method. For each iteration, show all intermediate calculations, including $\boldsymbol{w}$, $\boldsymbol{p}_i$, $\boldsymbol{r}_i$, ρ_i, $A\boldsymbol{p}_i$, and α.

(d) (10 pts) In our slides "sparse_CG2.pdf", at each iteration k of CG method, we need to solve the following optimization problem to find the search direction $\boldsymbol{p}_k$:

$$\min_{\boldsymbol{p}} \|\boldsymbol{p} - \boldsymbol{r}_{k-1}\|_2 \quad \text{s.t. } \boldsymbol{p} \in \operatorname{span}\{A\boldsymbol{p}_1, \ldots, A\boldsymbol{p}_{k-1}\}^{\perp} \tag{2}$$

Now, consider the linear system constructed in part (c) and the CG iteration with $k = 2$, where we have obtained $\boldsymbol{p}_1$, $\boldsymbol{r}_1$, and $\boldsymbol{p}_2$. Please solve the optimization problem (2) with $k = 2$ to find the solution $\boldsymbol{p}^*$, and give the relationship between $\boldsymbol{p}^*$ and $\boldsymbol{p}_2$.

Solution.

(a) We can calculate the gradient and Hessian as follows. Recall that

$$f(\boldsymbol{w}) = \frac{1}{2}\|(X\boldsymbol{w} - \boldsymbol{y})\|_2^2 = \frac{1}{2}(X\boldsymbol{w} - \boldsymbol{y})^T(X\boldsymbol{w} - \boldsymbol{y}) = \frac{1}{2}\boldsymbol{w}^T X^T X\boldsymbol{w} - \boldsymbol{y}^T X\boldsymbol{w} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}.$$

Since

$$\begin{aligned}
\boldsymbol{w}^T X^T X\boldsymbol{w} &= \sum_{i=1}^{m} w_i (X^T X\boldsymbol{w})_i \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i (X^T X)_{ij} w_j \\
&= w_1 (X^T X)_{11} w_1 + w_1 (X^T X)_{12} w_2 + \cdots + w_m (X^T X)_{mm} w_m.
\end{aligned}$$

Therefore

$$\frac{\partial}{\partial w_j}(\boldsymbol{w}^T X^T X\boldsymbol{w}) = 2((X^T X)_{j1} w_1 + (X^T X)_{j2} w_2 + \cdots + (X^T X)_{jm} w_m) = 2(X^T X\boldsymbol{w})_j.$$

For the term $\boldsymbol{y}^T X\boldsymbol{w}$, we have

$$\begin{aligned}
\boldsymbol{y}^T X\boldsymbol{w} &= \sum_{i=1}^{n} y_i (X\boldsymbol{w})_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} y_i X_{ij} w_j \\
&= y_1(X_{11}w_1 + X_{12}w_2 + \cdots + X_{1m}w_m) + \cdots + y_n(X_{n1}w_1 + X_{n2}w_2 + \cdots + X_{nm}w_m).
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial w_j}(\boldsymbol{y}^T X\boldsymbol{w}) = \sum_{i=1}^{n} y_i X_{ij} = (X^T\boldsymbol{y})_j.$$

Finally the gradient is

$$\nabla_{\boldsymbol{w}} f(\boldsymbol{w}) = \nabla_{\boldsymbol{w}}\left(\frac{1}{2}\boldsymbol{w}^T X^T X\boldsymbol{w} - \boldsymbol{y}^T X\boldsymbol{w} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}\right) = X^T X\boldsymbol{w} - X^T\boldsymbol{y}.$$

For Hessian matrix, since

$$\nabla_{\boldsymbol{w}} f(\boldsymbol{w}) = X^T X\boldsymbol{w} - X^T\boldsymbol{y},$$

we take the derivative with respect to $\boldsymbol{w}$ again and obtain

$$\nabla^2_{\boldsymbol{w}} f(\boldsymbol{w}) = \nabla_{\boldsymbol{w}}(X^T X\boldsymbol{w}) = X^T X.$$

(b) To minimize $f(\boldsymbol{w})$, we can set the gradient to zero and solve for $\boldsymbol{w}$.

$$\begin{aligned}
\nabla f(\boldsymbol{w}) = 0 &\Leftrightarrow X^T X\boldsymbol{w} - X^T\boldsymbol{y} = 0 \\
&\Leftrightarrow X^T X\boldsymbol{w} = X^T\boldsymbol{y}
\end{aligned}$$

Since the Hessian matrix $X^T X$ is invertible, we can get

$$\boldsymbol{w}^* = (X^T X)^{-1} X^T\boldsymbol{y}.$$

(c) Since

$$X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

we have

$$A = X^T X = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \text{ and } b = X^T\boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, the linear system become

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We use the Conjugate Gradient (CG) method to solve this linear system and obtain the solution of $\boldsymbol{w}$.

First we have

$$k = 0,\ \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \boldsymbol{r}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \rho_0 = \boldsymbol{r}^T \boldsymbol{r} = 2.$$

For $k = 1$, since

$$\sqrt{\rho_0} = \sqrt{2} > 0,$$

we calculate

$$\begin{aligned} \boldsymbol{p}_1 &= \boldsymbol{r}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A\boldsymbol{p}_1 = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \\ \alpha &= \frac{\rho_0}{\boldsymbol{p}_1^T A \boldsymbol{p}_1} = \frac{2}{15}. \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{w} + \alpha \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{2}{15} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/15 \\ 2/15 \end{bmatrix}, \\ \boldsymbol{r}_1 &= \boldsymbol{r}_0 - \alpha A \boldsymbol{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{2}{15} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}, \\ \rho_1 &= \boldsymbol{r}_1^T \boldsymbol{r}_1 = \frac{2}{9}. \end{aligned}$$

Next, when $k = 2$,

$$\sqrt{\rho_1} = \frac{\sqrt{2}}{3} > 0.$$

Thus, we calculate

$$\begin{aligned} \beta &= \frac{\rho_1}{\rho_0} = \frac{1}{9}, \\ \boldsymbol{p}_2 &= \boldsymbol{r}_1 + \beta \boldsymbol{p}_1 = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -2/9 \end{bmatrix}, \\ A\boldsymbol{p}_2 &= \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 4/9 \\ -2/9 \end{bmatrix} = \begin{bmatrix} 20/9 \\ -20/9 \end{bmatrix}, \\ \alpha &= \frac{\rho_1}{\boldsymbol{p}_2^T A \boldsymbol{p}_2} = \frac{2/9}{40/27} = \frac{3}{20}. \end{aligned}$$

Then,

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{w} + \alpha \boldsymbol{p}_2 = \begin{bmatrix} 2/15 \\ 2/15 \end{bmatrix} + \frac{3}{20} \begin{bmatrix} 4/9 \\ -2/9 \end{bmatrix} \\ &= \begin{bmatrix} 1/5 \\ 1/10 \end{bmatrix}, \\ \boldsymbol{r}_2 &= \boldsymbol{r}_1 - \alpha A \boldsymbol{p}_2 = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} - \frac{3}{20} \begin{bmatrix} 20/9 \\ -20/9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \rho_2 &= \boldsymbol{r}^T \boldsymbol{r} = 0. \end{aligned}$$

Since

$$\sqrt{\rho_2} = 0,$$

which satisfies the stopping condition.

Therefore, the CG method stops, and the solution of this linear system is

$$\boldsymbol{w} = \begin{bmatrix} 1/5 \\ 1/10 \end{bmatrix}.$$

(d) From the calculation results in part (c), we know that

$$\boldsymbol{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{p}_2 = \begin{bmatrix} 4/9 \\ -2/9 \end{bmatrix}, \boldsymbol{r}_1 = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}.$$

Since

$$A\boldsymbol{p}_1 = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix},$$

we need to find a vector $\boldsymbol{p}$ in

$$\operatorname{span}\{A\boldsymbol{p}_1\}^{\perp}.$$

Let

$$\boldsymbol{p} = \begin{bmatrix} s \\ t \end{bmatrix}$$

such that

$$\boldsymbol{p}^T A\boldsymbol{p}_1 = 0,$$

which implies

$$5s + 10t = 0.$$

This gives

$$s = -2t.$$

Hence, we can represent $\boldsymbol{p}$ as

$$\boldsymbol{p} = \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}.$$

Recall that

$$\begin{aligned} \|\boldsymbol{p} - \boldsymbol{r}_1\|_2 &= \left\| \begin{bmatrix} -2t \\ t \end{bmatrix} - \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} \right\|_2 \\ &= \sqrt{(-2t - \frac{1}{3})^2 + (t + \frac{1}{3})^2} \\ &= \sqrt{5t^2 + 2t + \frac{2}{9}} \\ &= \sqrt{5(t + \frac{1}{5})^2 + \frac{1}{45}} \end{aligned}$$

To minimize $\|\boldsymbol{p} - \boldsymbol{r}_1\|_2$, we take

$$t = -\frac{1}{5}.$$

Therefore,

$$\boldsymbol{p}^* = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix}.$$

Note that $\boldsymbol{p}^*$ is a scalar multiple of $\boldsymbol{p}_2$ obtained in part (c). Specifically,

$$\boldsymbol{p}^* = \frac{9}{10}\boldsymbol{p}_2.$$

Therefore, although the magnitudes of $\boldsymbol{p}^*$ and $\boldsymbol{p}_2$ are different, they represent the same search direction.

Problem 3 (40 pts). Consider the function

$$f(x) = x^2.$$

To approximate the function $f(x)$ using discrete Fourier transform, we will consider these $2m$ points

$$(x_0, f(x_0)), \dots, (x_{2m-1}, f(x_{2m-1}))$$

where

$$x_j = 1 + j, \qquad j = 0, \dots, 2m-1.$$

We wish to approximate the function using a Fourier series with $2n$ coefficients:

$$S_n(z) = \frac{a_0 + a_n \cos nz}{2} + \sum_{k=1}^{n-1} \left(a_k \cos kz + b_k \sin kz\right). \tag{1}$$

In this problem, we will consider the case where $m = n = 2$.

(a) (5 pts) Transform the coordinates x_j into equally spaced coordinates z_j on the interval $[-\pi, \pi)$ by using

$$x_j \mapsto z_j, \qquad z_j = -\pi + \frac{2\pi j}{2m}, \qquad j = 0, \dots, 2m-1.$$

Explicitly write the transformation formula between x and z. Then, calculate and list the values

$$(z_0, f(x_0)), \dots, (z_{2m-1}, f(x_{2m-1})).$$

(b) (10 pts) Give the matrices A_2, A_1, and P required by the fast Fourier transform algorithm. That is, the discrete Fourier transform matrix F can be decomposed into

$$F = A_2 A_1 P.$$

(c) (10 pts) Following (b), calculate the coefficient vector c given by

$$c = A_2 A_1 P y,$$

where $y_j = f(x_j)$. Then, convert the complex coefficients c_k into the Fourier coefficients in equation (1) by using

$$a_k = \operatorname{Re}\left(\frac{c_k(-1)^k}{m}\right), \qquad k = 0, 1, 2,$$

and

$$b_k = -\operatorname{Im}\left(\frac{c_k(-1)^k}{m}\right), \qquad k = 1.$$

Finally, write down the obtained Fourier series $S_2(z)$ in the form of equation (1).

(d) (10 pts) To verify the result obtained in part (c), directly calculate a_0, a_1, a_2, and b_1 using the formulas:

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos(kz_j), \qquad k = 0, 1, 2,$$

and

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin(kz_j).$$

Check that these values agree with the coefficients obtained from the FFT matrix method in part (c).

(e) (5 pts) The series we obtained in subproblem (c) approximates the shifted and scaled version of $f(x)$ in the interval $[-\pi, \pi)$. However, we are interested in approximating the original $f(x)$ in the interval $[1, 5)$. Given any $x \in [1, 5)$, show how to calculate the approximated value given by the Fourier series. That is, you need to rewrite the $S_n(z)$ obtained in subproblem (c) in terms of x.

Solution.

(a) Following the transformation specified in the problem, we map the $2m$ sample points to equally spaced points on the interval $[-\pi, \pi)$ by

$$z_j = -\pi + \frac{2\pi j}{2m}, \qquad j = 0, \ldots, 2m-1.$$

Thus,

$$\begin{aligned} z_0 &= -\pi, \\ z_1 &= -\pi + \frac{\pi}{2} = -\frac{\pi}{2}, \\ z_2 &= -\pi + \pi = 0, \\ z_3 &= -\pi + \frac{3\pi}{2} = \frac{\pi}{2}. \end{aligned}$$

Since $x_j = 1 + j$, we also have

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 4.$$

The relation between x and z is

$$z = \frac{x-3}{2}\pi.$$

Because $f(x) = x^2$, the function values are

$$\begin{aligned} f(x_0) &= 1^2 = 1, \\ f(x_1) &= 2^2 = 4, \\ f(x_2) &= 3^2 = 9, \\ f(x_3) &= 4^2 = 16. \end{aligned}$$

Therefore,

$$\begin{aligned} (z_0, f(x_0)) &= (-\pi, 1), \\ (z_1, f(x_1)) &= \left(-\frac{\pi}{2}, 4\right), \\ (z_2, f(x_2)) &= (0, 9), \\ (z_3, f(x_3)) &= \left(\frac{\pi}{2}, 16\right). \end{aligned}$$

(b) The δ we use is

$$\delta = e^{-i\pi/m} = e^{-i\pi/2} = -i.$$

To calculate A_2, we have

$$L = 2^2 = 4, \qquad r = \frac{2m}{L} = \frac{4}{4} = 1.$$

Therefore,

$$A_2 = I_r \otimes B_L = I_1 \otimes B_4 = \begin{bmatrix} I_2 & \Omega_2 \\ I_2 & -\Omega_2 \end{bmatrix},$$

where

$$\Omega_2 = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}.$$

Hence,

$$A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \delta \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\delta \end{bmatrix}. \tag{2}$$

To calculate A_1, we have

$$L = 2^1 = 2, \qquad r = \frac{2m}{L} = \frac{4}{2} = 2.$$

Therefore,

$$A_1 = I_r \otimes B_L = I_2 \otimes B_2.$$

Since

$$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

we get

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{3}$$

The permutation matrix P is calculated by reversing the binary representation of the indices:

$$\begin{aligned} 00 &\to 00, \\ 01 &\to 10, \\ 10 &\to 01, \\ 11 &\to 11. \end{aligned}$$

Therefore, the order becomes

$$0, 2, 1, 3.$$

Thus,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

(c) First, the vector y is

$$y = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix}.$$

By equation (4), we have

$$Py = \begin{bmatrix} 1 \\ 9 \\ 4 \\ 16 \end{bmatrix}.$$

Then, by equation (3),

$$A_1 P y = \begin{bmatrix} 1+9 \\ 1-9 \\ 4+16 \\ 4-16 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \\ 20 \\ -12 \end{bmatrix}.$$

Finally, by equation (2),

$$c = A_2 A_1 P y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \delta \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\delta \end{bmatrix} \begin{bmatrix} 10 \\ -8 \\ 20 \\ -12 \end{bmatrix}.$$

Since $\delta = -i$, we get

$$c = \begin{bmatrix} 10+20 \\ -8+\delta(-12) \\ 10-20 \\ -8-\delta(-12) \end{bmatrix} = \begin{bmatrix} 30 \\ -8+12i \\ -10 \\ -8-12i \end{bmatrix}.$$

From c, we calculate

$$a_k = \operatorname{Re}\left(\frac{c_k(-1)^k}{m}\right),$$

and

$$b_k = -\operatorname{Im}\left(\frac{c_k(-1)^k}{m}\right).$$

Since $m = 2$, we have

$$a_0 = \operatorname{Re}\left(\frac{c_0(-1)^0}{2}\right) = \operatorname{Re}\left(\frac{30}{2}\right) = 15.$$

Next,

$$a_1 = \operatorname{Re}\left(\frac{c_1(-1)^1}{2}\right) = \operatorname{Re}\left(\frac{(-8+12i)(-1)}{2}\right) = \operatorname{Re}(4-6i) = 4.$$

Also,

$$b_1 = -\operatorname{Im}\left(\frac{c_1(-1)^1}{2}\right) = -\operatorname{Im}(4-6i) = 6.$$

Finally,

$$a_2 = \operatorname{Re}\left(\frac{c_2(-1)^2}{2}\right) = \operatorname{Re}\left(\frac{-10}{2}\right) = -5.$$

Therefore,

$$a_0 = 15, \qquad a_1 = 4, \qquad a_2 = -5, \qquad b_1 = 6.$$

Thus, the transformed series in terms of z is

$$S_2(z) = \frac{a_0 + a_2 \cos 2z}{2} + a_1 \cos z + b_1 \sin z.$$

Substituting the coefficients, we obtain

$$S_2(z) = \frac{15 - 5\cos 2z}{2} + 4\cos z + 6 \sin z. \tag{5}$$

(d) We verify the coefficients obtained in part (c) by directly using the formulas for a_k and b_k.

From part (a), we have

$$z_0 = -\pi, \qquad z_1 = -\frac{\pi}{2}, \qquad z_2 = 0, \qquad z_3 = \frac{\pi}{2},$$

and

$$y = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix}.$$

Since $m = 2$, the coefficient formulas become

$$a_k = \frac{1}{2} \sum_{j=0}^{3} y_j \cos(kz_j), \qquad k = 0, 1, 2,$$

and

$$b_k = \frac{1}{2} \sum_{j=0}^{3} y_j \sin(kz_j).$$

First,

$$a_0 = \frac{1}{2} \sum_{j=0}^{3} y_j \cos(0z_j) = \frac{1}{2}(1 + 4 + 9 + 16) = 15.$$

Next,

$$a_1 = \frac{1}{2}\left[1\cos(-\pi) + 4\cos\left(-\frac{\pi}{2}\right) + 9\cos(0) + 16\cos\left(\frac{\pi}{2}\right)\right].$$

Because

$$\cos(-\pi) = -1, \qquad \cos\left(-\frac{\pi}{2}\right) = 0, \qquad \cos(0) = 1, \qquad \cos\left(\frac{\pi}{2}\right) = 0,$$

we have

$$a_1 = \frac{1}{2}\left[1(-1) + 4(0) + 9(1) + 16(0)\right] = 4.$$

For b_1,

$$b_1 = \frac{1}{2}\left[1\sin(-\pi) + 4\sin\left(-\frac{\pi}{2}\right) + 9\sin(0) + 16\sin\left(\frac{\pi}{2}\right)\right].$$

Because

$$\sin(-\pi) = 0, \qquad \sin\left(-\frac{\pi}{2}\right) = -1, \qquad \sin(0) = 0, \qquad \sin\left(\frac{\pi}{2}\right) = 1,$$

we have

$$b_1 = \frac{1}{2}\left[1(0) + 4(-1) + 9(0) + 16(1)\right] = 6.$$

Finally,

$$a_2 = \frac{1}{2}\left[1\cos(-2\pi) + 4\cos(-\pi) + 9\cos(0) + 16\cos(\pi)\right].$$

Because

$$\cos(-2\pi) = 1, \qquad \cos(-\pi) = -1, \qquad \cos(0) = 1, \qquad \cos(\pi) = -1,$$

we have

$$a_2 = \frac{1}{2}\left[1(1) + 4(-1) + 9(1) + 16(-1)\right] = -5.$$

Therefore, direct calculation gives

$$a_0 = 15, \qquad a_1 = 4, \qquad a_2 = -5, \qquad b_1 = 6.$$

These agree with the coefficients obtained from the FFT matrix method in part (c).

(e) Because

$$z = \frac{x-3}{2}\pi,$$

for any $x \in [1,5]$, we can calculate the approximated value by substituting

$$z = \frac{x-3}{2}\pi$$

into equation (5). Therefore,

$$S_2(x) = \frac{15 - 5\cos\left(2 \cdot \frac{x-3}{2}\pi\right)}{2} + 4\cos\left(\frac{x-3}{2}\pi\right) + 6\sin\left(\frac{x-3}{2}\pi\right).$$

Since

$$2 \cdot \frac{x-3}{2}\pi = (x-3)\pi,$$

we get

$$S_2(x) = \frac{15 - 5\cos\left((x-3)\pi\right)}{2} + 4\cos\left(\frac{x-3}{2}\pi\right) + 6\sin\left(\frac{x-3}{2}\pi\right).$$

Problem 4 (20 pts). In class, we learned about spline interpolation for the points and their function values

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)) \in (\mathbb{R}, \mathbb{R}).$$

In this problem, we extend the concept of splines to **geometric shape design**, specifically using B-splines to interpolate a set of control points

$$P_0, P_1, \ldots, P_n$$

in a 2D space $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, we have a B-spline curve $C(u)$ by considering P_0, P_1, P_2, P_3, P_4:

Let us learn about how to construct a B-spline on the following. A B-spline is a collection of **piecewise polynomial functions**

$$B_{i,k}(u) \text{ of degree } k$$

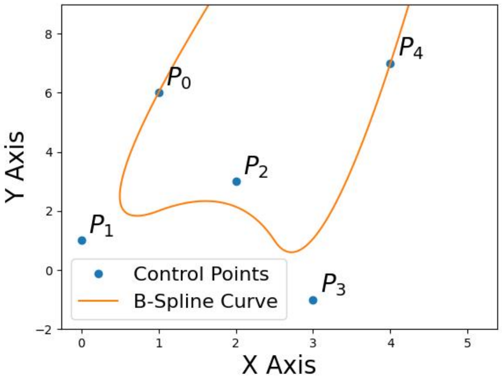

Figure 1: An example of clamped B-spline.

in a variable u, i.e.,

$$\begin{array}{ll} a_k^i u^k + a_{k-1}^i u^{k-1} + \cdots + a_1^i u + a_0^i, & \text{for some region } J_0^i, \\ b_k^i u^k + b_{k-1}^i u^{k-1} + \cdots + b_1^i u + b_0^i, & \text{for some region } J_1^i, \\ \vdots & \vdots \end{array}$$

for $i = 0, \ldots, n$, where n corresponds to the points $P_0, \ldots, P_n$. For each $i = 0, \ldots, n$, to construct $B_{i,\hat{k}}(u)$ of degree $\hat{k}$ efficiently and smoothly, we can utilize the Cox-de Boor recursion formula as follows. (Note that we will explain what the knots $u_0, \ldots, u_{n+k+1}$ are later).

- When $k = 0$,

$$B_{i,0}(u) = \begin{cases} 1, & \text{if } u_i \le u < u_{i+1} \\ 0, & \text{otherwise.} \end{cases}$$

- When $k = 1, \ldots, \hat{k}$,

$$B_{i,k}(u) = \begin{cases} \frac{u-u_i}{u_{i+k}-u_i} B_{i,k-1}(u) + \frac{u_{i+k+1}-u}{u_{i+k+1}-u_{i+1}} B_{i+1,k-1}(u), & \text{if } u_{i+k} \ne u_i \text{ and } u_{i+k+1} \ne u_{i+1}, \\ \frac{u-u_i}{u_{i+k}-u_i} B_{i,k-1}(u), & \text{if } u_{i+k} \ne u_i \text{ and } u_{i+k+1} = u_{i+1}, \\ \frac{u_{i+k+1}-u}{u_{i+k+1}-u_{i+1}} B_{i+1,k-1}(u), & \text{if } u_{i+k} = u_i \text{ and } u_{i+k+1} \ne u_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

After constructing the piecewise polynomial functions $B_{i,\hat{k}}(u)$, we can define the B-spline curve as

$$C(u) = \sum_{i=0}^{n} B_{i,\hat{k}}(u) P_i.$$

To define $C(u)$ on the points $P_0, \ldots, P_n$, we need to construct $B_{0,\hat{k}}(u), \ldots, B_{n,\hat{k}}(u)$, i.e., $n+1$ functions. Similarly, this construction is based on the functions $B_{0,\hat{k}-1}(u), \ldots, B_{n+1,\hat{k}-1}(u)$, and the number is $n+2$. In the end, we must have $B_{0,0}(u), \ldots, B_{n+k,0}(u)$, and those $n + k + 1$ functions require the information on

$$u_0, \ldots, u_{n+k+1}.$$

That is, we need to set the values of $u_0, \ldots, u_{n+k+1}$, which are also called knots.

In this problem, we consider the B-spline setting:

- the degree $\hat{k} = 2$,
- the control points P_0, P_1, P_2, P_3, P_4, and
- a knot vector $U = \{u_0, u_1, ..., u_7\} = \{0, 0, 0, 1, 2, 3, 3, 3\}$.

Please answer the following questions.

(a) (5 pts) Use Cox-de Boor recursion formula to find $B_{i,0}(u)$ for $i = 0, \ldots, 6$.

(b) (5 pts) Use Cox-de Boor recursion formula to find $B_{i,1}(u)$ for $i = 0, \ldots, 5$.

(c) (5 pts) Based on the B-spline setting defined above, now you have $B_{i,2}$ for $i = 0, \ldots, 4$ as follows (you may use your result in (a) and (b) to verify these).

$$B_{0,2}(u) = (1-u)^2 B_{2,0}(u)$$

$$B_{1,2}(u) = \left[u(1-u) + \frac{u(2-u)}{2}\right] B_{2,0}(u) + \frac{(2-u)^2}{2} B_{3,0}(u)$$

$$B_{2,2}(u) = \frac{u^2}{2} B_{2,0}(u) + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2}\right] B_{3,0}(u) + \frac{(3-u)^2}{2} B_{4,0}(u)$$

$$B_{3,2}(u) = \frac{(u-1)^2}{2} B_{3,0}(u) + \left[\frac{(u-1)(3-u)}{2} + (3-u)(u-2)\right] B_{4,0}(u)$$

$$B_{4,2}(u) = (u-2)^2 B_{4,0}(u)$$

Find the B-spline curve $C(u)$ in terms of u and P_i for $i = 0, \ldots, 4$, under

- $0 \le u < 1$,
- $1 \le u < 2$, and
- $2 \le u < 3$.

(d) (2 pts) Use the result of (c) to verify whether the curve $C(u)$ passes through P_0 and P_4.

(e) (3 pts) Please find the corresponding control points with respect to the curve segments in the following table.

Curve segments	Corresponding control points
$C(u)$, where $0 \le u < 1$	
$C(u)$, where $1 \le u < 2$	
$C(u)$, where $2 \le u < 3$	

That is, if we modify a control point, we may only change part of the curve, i.e., affecting the curve locally.

Solution.

(a) Given the knot vector $U = \{0, 0, 0, 1, 2, 3, 3, 3\}$, the individual knot values are:

$$u_0 = 0, \quad u_1 = 0, \ldots, u_5 = 3, \quad u_6 = 3, \quad u_7 = 3$$

By the definition of Cox-de Boor recursion formula, we have:

$$\begin{aligned}
&B_{0,0}(u) = 0 \quad \text{(empty interval } [0, 0))\\
&B_{1,0}(u) = 0 \quad \text{(empty interval } [0, 0))\\
&B_{2,0}(u) = \begin{cases} 1, & \text{if } 0 \le u < 1 \\ 0, & \text{otherwise} \end{cases}\\
&B_{3,0}(u) = \begin{cases} 1, & \text{if } 1 \le u < 2 \\ 0, & \text{otherwise} \end{cases}\\
&B_{4,0}(u) = \begin{cases} 1, & \text{if } 2 \le u < 3 \\ 0, & \text{otherwise} \end{cases}\\
&B_{5,0}(u) = 0 \quad \text{(empty interval } [3, 3))\\
&B_{6,0}(u) = 0 \quad \text{(empty interval } [3, 3))
\end{aligned}$$

(b) Applying the Cox-de Boor recursion formula for $k = 1$:

$$B_{i,1}(u) = \frac{u - u_i}{u_{i+1} - u_i} B_{i,0}(u) + \frac{u_{i+2} - u}{u_{i+2} - u_{i+1}} B_{i+1,0}(u)$$

So we have to consider u_i, u_{i+1}, u_{i+2}, then

$$\begin{aligned}
&u_0 = 0, u_1 = 0, u_2 = 0 \Rightarrow B_{0,1}(u) = 0\\
&u_1 = 0, u_2 = 0, u_3 = 1 \Rightarrow B_{1,1}(u) = \frac{1 - u}{1 - 0} B_{2,0} = (1 - u)B_{2,0}\\
&u_2 = 0, u_3 = 1, u_4 = 2 \Rightarrow B_{2,1}(u) = \frac{u - 0}{1 - 0} B_{2,0} + \frac{2 - u}{2 - 1} B_{3,0} = uB_{2,0} + (2 - u)B_{3,0}\\
&u_3 = 1, u_4 = 2, u_5 = 3 \Rightarrow B_{3,1}(u) = \frac{u - 1}{2 - 1} B_{3,0} + \frac{3 - u}{3 - 2} B_{4,0} = (u - 1)B_{3,0} + (3 - u)B_{4,0}\\
&u_4 = 2, u_5 = 3, u_6 = 3 \Rightarrow B_{4,1}(u) = \frac{u - 2}{3 - 2} B_{4,0} = (u - 2)B_{4,0}\\
&u_5 = 3, u_6 = 3, u_7 = 3 \Rightarrow B_{5,1}(u) = 0
\end{aligned}$$

(c) By the definition of B-spline curve function, we have

$$C(u) = \sum_{i=0}^{4} B_{i,2}(u)P_i = B_{0,2}(u)P_0 + B_{1,2}(u)P_1 + B_{2,2}(u)P_2 + B_{3,2}(u)P_3 + B_{4,2}(u)P_4$$

Then we can derive the piecewise polynomial equations:
For the interval $0 \le u < 1$, we have

$$\begin{aligned}
C(u) &= P_0(1 - u)^2 + P_1 \left[u(1 - u) + \frac{u(2 - u)}{2} \right] + P_2 \frac{u^2}{2}\\
&= P_0(1 - u)^2 + P_1 \frac{4u - 3u^2}{2} + P_2 \frac{u^2}{2}
\end{aligned}$$

For the interval $1 \leq u < 2$, we have

$$\begin{aligned}
C(u) &= P_1 \frac{(2-u)^2}{2} + P_2 \left[\frac{2u-u^2}{2} + \frac{4u-3-u^2}{2} \right] + P_3 \frac{(u-1)^2}{2} \\
&= P_1 \frac{(2-u)^2}{2} + P_2 \frac{-2u^2+6u-3}{2} + P_3 \frac{(u-1)^2}{2}
\end{aligned}$$

For the interval $2 \leq u < 3$, we have

$$\begin{aligned}
C(u) &= P_2 \frac{(3-u)^2}{2} + P_3 \left[\frac{4u-3-u^2}{2} + (-u^2+5u-6) \right] + P_4 (u-2)^2 \\
&= P_2 \frac{(3-u)^2}{2} + P_3 \frac{-3u^2+14u-15}{2} + P_4 (u-2)^2
\end{aligned}$$

(d) For P_0, we consider $C(u)$ under $0 \leq u < 1$, substitute $u = 0$ into $C(u)$,

$$C(0) = P_0 (1-0)^2 + P_1 \frac{4 \times 0 - 3 \times 0^2}{2} + P_2 \frac{0^2}{2} = P_0$$

Hence, $C(u)$ pass through P_0.

For P_4, we accept both answers here.

- $C(u)$ does not pass through P_4: Since $C(u)$ does not cover the case where $u = 3$, the curve $C(u)$ does not pass through P_4.
- $C(u)$ passes through P_4: We consider $C(u)$ under $2 \leq u < 3$ and take the left hand limit to approach the boundary:

  $$\lim_{u \to 3^-} C(u) = \lim_{u \to 3^-} P_2 \frac{(3-u)^2}{2} + P_3 \frac{-3u^2+14u-15}{2} + P_4 (u-2)^2 = P_2 \times 0 + P_3 \times 0 + P_4 \times (3-2)^2 = P_4.$$

  The left hand limit shows that the curve approaches P_4 as u approaches 3 from the left. Although P_4 is not included in the curve, it can be viewed as the endpoint of the curve segment.

Note that you need to give the explanation for getting the scores.

(e) By the result of (c), we can know that

Curve segment	Control points
$0 \leq u < 1$	P_0, P_1, P_2
$1 \leq u < 2$	P_1, P_2, P_3
$2 \leq u < 3$	P_2, P_3, P_4