Relative error I

- Usually calculating

$$\frac{|\hat{\alpha} - \alpha|}{|\alpha|}$$

is not practical because α is unknown

- A more reasonable estimate is

$$\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|}$$

Relative error II

- If

$$\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|} \leq 0.1,$$

then

$$\frac{1}{1.1}\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|} \leq \frac{|\hat{\alpha} - \alpha|}{|\alpha|} \leq \frac{1}{0.9}\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|}$$

Proof:

$$|\alpha| - |\hat{\alpha}| \leq |\hat{\alpha} - \alpha| \leq 0.1|\hat{\alpha}|$$
$$|\alpha| \leq 1.1|\hat{\alpha}|$$

Relative error III

Similarly,

$$|\hat{\alpha}| - |\alpha| \leq 0.1|\hat{\alpha}|$$
$$|\alpha| \geq 0.9|\hat{\alpha}|$$

Condition of a Linear System I

- Solving a linear system

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix}, \text{ solution } = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Right-hand side slightly modified

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{bmatrix}, \text{ solution } = \begin{bmatrix} 9.2 \\ -12.6 \\ 4.5 \\ -1.1 \end{bmatrix}$$

Condition of a Linear System II

A small modification causes a huge error

- Matrix slightly modified

$$\begin{bmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.89 & 9 \\ 6.99 & 4.99 & 9 & 9.98 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix}$$

$$\text{solution} = \begin{bmatrix} -81 \\ 137 \\ -34 \\ 22 \end{bmatrix}$$

Condition of a Linear System III

- Right-hand side slightly modified

$$Ax = b$$
$$A(x + \delta x) = b + \delta b$$
$$\delta x = A^{-1}\delta b \Rightarrow \|\delta x\| \leq \|A^{-1}\|\|\delta b\|$$
$$\|b\| \leq \|A\|\|x\|$$
$$\frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

Condition of a Linear System IV

- Matrix modified

$$
\begin{aligned}
Ax &= b \\
(A + \delta A)(x + \delta x) &= b \\
Ax + A\delta x + \delta A(x + \delta x) &= b \\
\delta x &= -A^{-1}\delta A(x + \delta x) \\
\|\delta x\| &\leq \|A^{-1}\|\|\delta A\|\|x + \delta x\| \\
\frac{\|\delta x\|}{\|x + \delta x\|} &\leq \|A\|\|A^{-1}\|\frac{\|\delta A\|}{\|A\|}
\end{aligned}
$$

Condition of a Linear System V

Note that here we can estimate only

$$\frac{\|\delta x\|}{\|x + \delta x\|} \text{ instead of } \frac{\|\delta x\|}{\|x\|};$$

see the discussion on relative errors earlier

- Clearly, error is strongly related to $\|A\|\|A^{-1}\|$, which is defined as the condition of A
- A smaller condition number is better