Vector and Matrix Norms I

- We have different types of objects: scalar, vector, matrix

  How to calculate errors?
- Scalar:

  absolute error:
$$|\hat{\alpha} - \alpha|$$
  relative error:
$$\frac{|\hat{\alpha} - \alpha|}{|\alpha|}$$
- Vectors: vector norm

Vector and Matrix Norms II

Norm is a way to calculate the length of a vector

- l-norm:

$$\|x\|_l = \sqrt[l]{|x_1|^l + \cdots + |x_n|^l}$$

- 1-norm:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Vector and Matrix Norms III

- ∞-norm: it is defined as

$$\|x\|_\infty \equiv \lim_{l\to\infty} \|x\|_l$$

We can then do the following calculation

$$\begin{aligned}\lim_{l\to\infty} \|x\|_l &= \lim_{l\to\infty} \sqrt[l]{|x_1|^l + \cdots + |x_n|^l} \\ &= \max |x_i|\end{aligned}$$

Proof:

$$\sqrt[l]{(\max |x_i|)^l} \leq \sqrt[l]{|x_1|^l + \cdots + |x_n|^l} \leq \sqrt[l]{n(\max |x_i|)^l}$$

Vector and Matrix Norms IV

- Example:

$$x = \begin{bmatrix} 1 \\ 100 \\ 9 \end{bmatrix} \text{ and } \hat{x} = \begin{bmatrix} 1.1 \\ 99 \\ 11 \end{bmatrix}$$

$$\|\hat{x} - x\|_\infty = 2, \frac{\|\hat{x}-x\|_\infty}{\|x\|_\infty} = 0.02, \frac{\|\hat{x}-x\|_\infty}{\|\hat{x}\|_\infty} = 0.0202$$

$$\|\hat{x} - x\|_2 = 2.238, \frac{\|\hat{x}-x\|_2}{\|x\|_2} = 0.0223, \frac{\|\hat{x}-x\|_2}{\|\hat{x}\|_2} = 0.0225$$

- For l-norm, we say that all norms are equivalent

Vector and Matrix Norms V

- For l_1 and l_2 norms, there exist c_1 and c_2 such that

$$c_1\|x\|_{l_1} \leq \|x\|_{l_2} \leq c_2\|x\|_{l_1}$$

  for all $x \in R^n$
- Example:

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \tag{1}$$
$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty \tag{2}$$
$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty \tag{3}$$

  Proofs are omitted as they are easy

Vector and Matrix Norms VI

- Therefore, you can just choose a norm for your convenience
- Matrix norm: How to define the distance between two matrices?
- Usually a norm should satisfy

$$
\begin{aligned}
\|A\| &\geq 0 \\
\|A+B\| &\leq \|A\| + \|B\| \\
\|\alpha A\| &= |\alpha| \|A\|,
\end{aligned} \tag{4}
$$

where α is a scalar

Vector and Matrix Norms VII

- Definition:

$$\|A\|_l \equiv \max_{x \neq 0} \frac{\|Ax\|_l}{\|x\|_l} = \max_{\|x\|_l = 1} \|Ax\|_l$$

- Proof of (4)

$$\|A + B\| = \max_{\|x\|=1} \|(A + B)x\|$$

$$\leq \max_{\|x\|=1} (\|Ax\| + \|Bx\|) \leq \max_{\|x\|=1} \|Ax\| + \max_{\|x\|=1} \|Bx\|$$

Vector and Matrix Norms VIII

- From this definition,

$$\frac{\|Ax\|}{\|x\|} \leq \|A\|, \forall x$$

so

$$\|Ax\| \leq \|A\|\|x\|$$