Newton Method: Quadratic Convergence I

- Use Newton method to solve

$$f(x) = 0$$

- Assume f satisfies that
    1. f is continuously differentiable
    2. f' is Lipschitz continuous:

$$|f'(y) - f'(x)| \leq \alpha |y - x|, \forall x, y$$

where $\alpha > 0$ is the Lipschitz constant

Newton Method: Quadratic Convergence II

Theorem 1

If $\{x^k\} \to x^*$ *and* $f'(x^*) \neq 0$*, then*

1. $f(x^*) = 0$
2. $\exists L \geq 1, \delta > 0$ *such that* $\forall k \geq L$

$$|x^{k+1} - x^*| \leq \delta |x^k - x^*|^2$$

Proof

Newton Method: Quadratic Convergence III

From the update rule and Lemma 3,

$$\begin{aligned} f(x^{k+1}) &= f(x^k) + f'(x^k)(x^{k+1} - x^k) + e(x^{k+1}, x^k) \\ &= 0 + e(x^{k+1}, x^k), \end{aligned}$$

where

$$|e(x^{k+1}, x^k)| \le \frac{1}{2}\alpha |x^{k+1} - x^k|^2$$

Thus

$$|f(x^{k+1})| \le \frac{1}{2}\alpha |x^{k+1} - x^k|^2.$$

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Since $\{x^k\} \to x^*$

$$|f(x^{k+1})| \to 0$$

Since f is continuous,

$$f(x^{k+1}) \to f(x^*) \text{ implies } f(x^*) = 0.$$

Define

$$\bar{\beta} \equiv |f'(x^*)^{-1}|.$$

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Since

$$|x^k - x^*| \to 0$$

and f' is Lipschitz continuous, $\exists L \geq 1$ such that $\forall k \geq L$,

$$\begin{aligned} |f'(x^*)^{-1}(f'(x^k) - f'(x^*))| &\leq |f'(x^*)^{-1}||f'(x^k) - f'(x^*)| \\ &\leq \alpha\bar{\beta}|x^k - x^*| \\ &\leq \frac{1}{2}. \end{aligned}$$

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For $k \geq L$, from Lemma 2, $f'(x^k)^{-1}$ exists and

$$|f'(x^k)^{-1}| \leq 2\bar{\beta} \tag{1}$$

By the definition of Newton method:

$$x^{k+1} - x^* = x^k - x^* - f'(x^k)^{-1} f(x^k) \tag{2}$$

Since $f(x^*) = 0$

$$f(x^*) = f(x^k) + f'(x^k)(x^* - x^k) + e(x^*, x^k) = 0$$

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Then multiply $f'(x^k)^{-1}$ on each term:

$$f'(x^k)^{-1}f(x^k) = x^k - x^* - f'(x^k)^{-1}e(x^*, x^k) \quad (3)$$

Thus (2) and (3) imply

$$x^{k+1} - x^* = f'(x^k)^{-1}e(x^*, x^k) \quad (4)$$

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From (1), (4), and Lemma 3, for $k \geq L$,

$$\begin{aligned} |x^{k+1} - x^*| &\leq |f'(x^k)^{-1}||e(x^*, x^k)| \\ &\leq 2\bar{\beta}\frac{1}{2}\alpha|x^k - x^*|^2 \\ &= \delta|x^k - x^*|^2, \end{aligned}$$

where we define

$$\delta = \bar{\beta}\alpha$$

Then the proof is complete

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Lemma 2

If

$$|f'(x^*)^{-1}(f'(x^k) - f'(x^*))| < 1 \quad (5)$$

then

$$|f'(x^k)^{-1}| \le \frac{|f'(x^*)^{-1}|}{1 - |f'(x^*)^{-1}(f'(x^k) - f'(x^*))|} \quad (6)$$

Proof

Newton Method: Quadratic Convergence X

The inequality (6)

$$|f'(x^k)^{-1}| \leq \frac{|f'(x^*)^{-1}|}{1 - |f'(x^*)^{-1} f'(x_k) - 1|}$$

is equivalent to

$$\begin{aligned} & |f'(x_k)^{-1}| - |f'(x^*)^{-1} - f'(x_k)^{-1}| \\ = \; & |f'(x_k)^{-1}| - |f'(x_k)^{-1} - f'(x^*)^{-1}| \\ \leq \; & |f'(x^*)^{-1}|, \end{aligned} \tag{7}$$

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where the last inequality is from the property of absolute values

Note that we need (5) only for ensuring that the denominator of the right-hand side of (6) is positive.

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Lemma 3

If f' is Lipschitz continuous with constant $\alpha > 0$ and

$$f(y) = f(x) + f'(x)(y - x) + e(y, x),$$

then $\forall x, y$

$$|e(y, x)| \leq \frac{1}{2}\alpha|y - x|^2$$

Proof

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Because

$$
\begin{aligned}
& \frac{d(f(x+t(y-x))}{dt} \\
= \; & f'(x+t(y-x))(y-x),
\end{aligned}
$$

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we have

$$
\begin{aligned}
& \int_0^1 (f'(x + t(y-x)) - f'(x))(y-x)dt \\
= & \left(\int_0^1 f'(x + t(y-x))(y-x)dt \right) - f'(x)(y-x) \\
= & f(x + t(y-x)) \Big|_{t=0}^{1} - f'(x)(y-x) \\
= & f(y) - f(x) - f'(x)(y-x)
\end{aligned}
$$

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Then

$$
\begin{aligned}
& |e(y,x)| \\
\le & \int_0^1 |(f'(x+t(y-x)) - f'(x))(y-x)| dt \qquad (8) \\
\le & \int_0^1 \alpha |(x+t(y-x)) - x||(y-x)| dt \qquad (9) \\
\le & \int_0^1 \alpha t |(y-x)|^2 dt = \frac{1}{2}\alpha |y-x|^2,
\end{aligned}
$$

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where (8) to (9) is from the Lipschitz condition