Iterative Methods for Solving Linear Systems I

- Materials are mainly from chapter 10 of the book Matrix Computation
- An iterative process:

$$x_1, x_2, \ldots, \rightarrow x^*$$

- We hope

$$Ax^* = b$$

- Gaussian elimination is $O(n^3)$

Iterative Methods for Solving Linear Systems II

If $x_k \to x_{k+1}$ takes $O(n^r)$, l iterations, and $n^r l < n^3$, iterative methods can be faster

- Accuracy and sparsity are other considerations

Jacobi and Gauss-Seidel Methods I

- A three by three system $Ax = b$

$$
\begin{aligned}
x_1 &= (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11} \\
x_2 &= (b_2 - a_{21}x_1 - a_{23}x_3)/a_{22} \\
x_3 &= (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33}
\end{aligned}
$$

- Jacobi method to update from x_k to x_{k+1}

$$
\begin{aligned}
(x_{k+1})_1 &= (b_1 - a_{12}(x_k)_2 - a_{13}(x_k)_3)/a_{11} \\
(x_{k+1})_2 &= (b_2 - a_{21}(x_k)_1 - a_{23}(x_k)_3)/a_{22} \\
(x_{k+1})_3 &= (b_3 - a_{31}(x_k)_1 - a_{32}(x_k)_2)/a_{33}
\end{aligned}
$$

Jacobi and Gauss-Seidel Methods II

- The general case

  for i = 1:n

  $$(x_{k+1})_i = (b_i - \sum_{j=1}^{i-1} a_{ij}(x_k)_j - \sum_{j=i+1}^{n} a_{ij}(x_k)_j)/a_{ii}$$

  end
- Gauss-Seidel iteration

$$\begin{aligned}
(x_{k+1})_1 &= (b_1 - a_{12}(x_k)_2 - a_{13}(x_k)_3)/a_{11} \\
(x_{k+1})_2 &= (b_2 - a_{21}(x_{k+1})_1 - a_{23}(x_k)_3)/a_{22} \\
(x_{k+1})_3 &= (b_3 - a_{31}(x_{k+1})_1 - a_{32}(x_{k+1})_2)/a_{33}
\end{aligned}$$

- The general case

Jacobi and Gauss-Seidel Methods III

for i = 1:n

$(x_{k+1})_i =$

$$(b_i - \textstyle\sum_{j=1}^{i-1} a_{ij}(x_{k+1})_j - \sum_{j=i+1}^{n} a_{ij}(x_k)_j)/a_{ii}$$

end

- The iterates may diverge

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, \text{sol} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Jacobi and Gauss-Seidel Methods IV

determinant $= 1 + 8 + 8 - 4 - 4 - 4 \neq 0$, so $A^{-1}b$ exists. However,

$$x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, x_1 = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, x_2 = \begin{bmatrix} 5 - 10 - 10 \\ 5 - 10 - 10 \\ 5 - 10 - 10 \end{bmatrix} = \begin{bmatrix} -15 \\ -15 \\ -15 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 5 + 30 + 30 \\ 5 + 30 + 30 \\ 5 + 30 + 30 \end{bmatrix} = \begin{bmatrix} 65 \\ 65 \\ 65 \end{bmatrix}$$

- Convergence analysis

Jacobi and Gauss-Seidel Methods V

When does the method eventually go to a solution?

$$
\begin{aligned}
a_{11}x_1 &= b_1 - a_{12}x_2 - a_{13}x_3 \\
a_{22}x_2 &= b_2 - a_{21}x_1 - a_{23}x_3 \\
a_{33}x_3 &= b_3 - a_{31}x_1 - a_{32}x_2
\end{aligned}
$$

This is like

$$
\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b
$$

Jacobi and Gauss-Seidel Methods VI

If

$$M = \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix}$$

then

$$A = M - N \text{ and } Mx_{k+1} = Nx_k + b$$

Jacobi and Gauss-Seidel Methods VII

- Spectral radius:

$$\rho(A) = \max\{|\lambda| \mid \lambda \in \lambda(A)\},$$

where $\lambda(A)$ contains all eigenvalues of A

Jacobi and Gauss-Seidel Methods VIII

Theorem

Assume

$$A = M - N,$$

where

$$A, M \text{ are non-singular and } \rho(M^{-1}N) < 1.$$

Then

$$Mx_{k+1} = Nx_k + b$$

leads to the convergence of $\{x_k\}$ *to* $A^{-1}b$ *for any starting vector* x_1

Jacobi and Gauss-Seidel Methods IX

Proof:

$$
\begin{aligned}
&Ax = b \Rightarrow Mx = Nx + b \\
&M(x_{k+1} - x) = N(x_k - x) \\
&x_{k+1} - x = M^{-1}N(x_k - x) \\
&x_{k+1} - x = (M^{-1}N)^k(x_1 - x) \\
&\rho(M^{-1}N) < 1 \Rightarrow (M^{-1}N)^k \to 0 \\
&x_{k+1} - x \to 0
\end{aligned}
$$

Jacobi and Gauss-Seidel Methods X

- Reasons why

$$\rho(M^{-1}N) < 1 \Rightarrow (M^{-1}N)^k \to 0?$$

This is quite complicated, so we omit the derivation here.