Cholesky Factorization I

- Another implementation: treat

$$A = LL^T$$

as a system of equations with unknowns L and solve it.

$$A_{ij} = \sum_{k=1}^{n} L_{ik}(L^T)_{kj} = \sum_{k=1}^{n} L_{ik}L_{jk} = \sum_{k=1}^{n} L_{jk}L_{ik}$$

$$A_{:,j} = \sum_{k=1}^{n} L_{jk}L_{:,k}$$

Cholesky Factorization II

Because L is lower triangular

$$L_{j,j+1} = \cdots = L_{j,n} = 0$$

Thus

$$A_{:,j} = \sum_{k=1}^{j} L_{jk} L_{:,k}$$

$$L_{jj} L_{:,j} = A_{:,j} - \sum_{k=1}^{j-1} L_{jk} L_{:,k}$$

right-hand side: involve 1st to $(j-1)$st columns

Cholesky Factorization III

left-hand side: jth column

- jth column unknown, 1st to $(j-1)$st columns known.
- Let

$$L_{jj}L_{jj} = A_{j,j} - \sum_{k=1}^{j-1} L_{jk}L_{j,k}$$

Then

$$L_{j+1:n,j} = (A_{j+1:n,j} - \sum_{k=1}^{j-1} L_{jk}L_{j+1:n,k})/L_{jj}$$

Cholesky Factorization IV

In practical implementation we calculate

$$A_{j:n,j} - \sum_{k=1}^{j-1} L_{jk} L_{j:n,k}$$

first. Code:

```
for j=1:n
    v(j:n) = A(j:n,j)
    for k=1:j-1
        v(j:n) = v(j:n) - L(j,k)L(j:n,k)
    end
```

Cholesky Factorization V

```
        L(j:n,j) = v(j:n)/sqrt(v(j))
    end
```

- Note that

```
v(j:n) = v(j:n) - L(j,k)L(j:n,k)
```

  is a axpy operation

Cholesky Factorization VI

- Operations: a 3-level for loop

$$
\begin{aligned}
& \sum_{j=1}^{n} \sum_{k=1}^{j-1} 2(n-j+1) \\
= \; & 2\sum_{j=1}^{n} (j-1)(n-j+1) \\
= \; & 2\sum_{j=1}^{n} (nj - n - j^2 + j + j - 1) \\
\approx \; & 2(n\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}) = O(\frac{n^3}{3})
\end{aligned}
$$

- Cholesky factorization: half operations of LU factorization