Special Matrices and Cholesky Factorization I

- There are many special matrices. For example, diagonal, tri-diagonal, banded matrices, positive definite matrices, etc.
- A banded matrix

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 4 & 5 & 6 & 7 & 0 & 0 \\ 8 & 9 & 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 & 6 \end{bmatrix}$$

Special Matrices and Cholesky Factorization II

- Many applications involve symmetric positive definite matrices
- Definition: An $n \times n$ matrix A is positive definite if $x^T A x > 0, \forall x \in R^n$, $x \neq 0$
- A is symmetric positive definite if and only if all A's eigenvalues are positive
- A is SPD $\Rightarrow$ all diagonal elements are positive, all principle sub-matrices are positive definite

Special Matrices and Cholesky Factorization III

- A is symmetric positive definite (SPD) $\Rightarrow$ there is a lower triangular matrix L such that

$$A = LL^T$$

$\Rightarrow$ Cholesky factorization

Cholesky Factorization I

- We have

$$
\begin{aligned}
A &= \begin{bmatrix} \alpha & v^T \\ v & B \end{bmatrix} \qquad (\alpha > 0) \\
&= \begin{bmatrix} \sqrt{\alpha} & 0 \\ v/\sqrt{\alpha} & I \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & v^T/\sqrt{\alpha} \\ 0 & B - \frac{vv^T}{\alpha} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{\alpha} & 0 \\ v/\sqrt{\alpha} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B - \frac{vv^T}{\alpha} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & v^T/\sqrt{\alpha} \\ 0 & I \end{bmatrix}
\end{aligned}
$$

Cholesky Factorization II

- If

$$B - \frac{vv^T}{\alpha} = \bar{L}\bar{L}^T,$$

then

$$\begin{aligned}
& \begin{bmatrix} \sqrt{\alpha} & 0 \\ v/\sqrt{\alpha} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \bar{L}\bar{L}^T \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & v^T/\sqrt{\alpha} \\ 0 & I \end{bmatrix} \\
= \; & \left(\begin{bmatrix} \sqrt{\alpha} & 0 \\ v/\sqrt{\alpha} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \bar{L} \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & \bar{L}^T \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & v^T/\sqrt{\alpha} \\ 0 & I \end{bmatrix} \right) \\
= \; & \begin{bmatrix} \sqrt{\alpha} & 0 \\ v/\sqrt{\alpha} & \bar{L} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & v^T/\sqrt{\alpha} \\ 0 & \bar{L}^T \end{bmatrix}.
\end{aligned}$$

Cholesky Factorization III

If we can prove $B - \frac{vv^T}{\alpha}$ is also PD, the same procedure could be applied to $B - \frac{vv^T}{\alpha}$.

- If $A = \begin{bmatrix} \alpha & v^T \\ v & B \end{bmatrix}$ is PD, $\Rightarrow B - \frac{vv^T}{\alpha}$ is PD.

For any $x \neq 0$

$$
\begin{aligned}
& x^T(B - \frac{vv^T}{\alpha})x = x^T Bx - x^T(\frac{vv^T}{\alpha})x \\
= & \begin{bmatrix} -\frac{v^Tx}{\alpha} & x^T \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{v^Tx}{\alpha}v + Bx \end{bmatrix} \\
= & \begin{bmatrix} -\frac{v^Tx}{\alpha} & x^T \end{bmatrix} \begin{bmatrix} \alpha & v^T \\ v & B \end{bmatrix} \begin{bmatrix} -\frac{v^Tx}{\alpha} \\ x \end{bmatrix} > 0
\end{aligned}
$$

Cholesky Factorization IV

- Implementation: the outer product form

$$A = \begin{bmatrix} \alpha & v^T \\ v & B \end{bmatrix} (\alpha > 0)$$
$$= \begin{bmatrix} \sqrt{\alpha} & 0 \\ v/\sqrt{\alpha} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B - \frac{vv^T}{\alpha} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & v^T/\sqrt{\alpha} \\ 0 & I \end{bmatrix}$$

```
for k=1:n
    A(k,k) = sqrt(A(k,k))
    A(k+1:n,k) = A(k+1:n,k)/A(k,k)
    for j=k+1:n
        A(j:n,j) = A(j:n,j) -
```

Cholesky Factorization V

```
            A(j:n,k)A(j,k)
        end
    end
```

The inner loop comes from

$$B_{:,j} - (v/\sqrt{\alpha})_j \frac{v}{\sqrt{\alpha}}$$