Pivoting: Avoid Small Pivots I

- Pivoting in LU:

$$A = \begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 18 & -12 \end{bmatrix}$$

Largest absolute value in the first column: 6

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Pivoting: Avoid Small Pivots II

Then

$$P_1 A = \begin{bmatrix} 6 & 18 & -12 \\ 2 & 4 & -2 \\ 3 & 17 & 10 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$M_1 P_1 A = \begin{bmatrix} 6 & 18 & -12 \\ 0 & -2 & 2 \\ 0 & 8 & 16 \end{bmatrix}$$

Pivoting: Avoid Small Pivots III

- Swap 2nd and 3rd rows:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/4 & 1 \end{bmatrix}$$

$$M_2 P_2 M_1 P_1 A = \begin{bmatrix} 6 & 18 & -12 \\ 0 & 8 & 16 \\ 0 & 0 & 6 \end{bmatrix}$$

- The general form

$$M_{n-1} P_{n-1} \cdots M_1 P_1 A = U$$

Pivoting: Avoid Small Pivots IV

- Now we have

$$Ax = b$$
$$(M_{n-1}P_{n-1} \cdots M_1 P_1)Ax = (M_{n-1}P_{n-1} \cdots M_1 P_1)b$$

so we can solve

$$Ux = (M_{n-1}P_{n-1} \cdots M_1 P_1)b$$

- But how to store $M_1, P_1, \ldots, M_{n-1}, P_{n-1}$?
- So we don't have a practical procedure yet

Pivoting: Practical Procedure I

- Let's try the following procedure by overwriting A

$$\begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 18 & -12 \end{bmatrix} \qquad p(1) = 3$$

$$\begin{bmatrix} 6 & 18 & -12 \\ 2 & 4 & -2 \\ 3 & 17 & 10 \end{bmatrix} \qquad \text{swap two rows}$$

- Then

$$\begin{bmatrix} 6 & 18 & -12 \\ 1/3 & -2 & 2 \\ 1/2 & 8 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 18 & -12 \\ 1/2 & 8 & 16 \\ 1/3 & -2 & 2 \end{bmatrix} \qquad p(2) = 3$$

$$\begin{bmatrix} 6 & 18 & -12 \\ 1/2 & 8 & 16 \\ 1/3 & -1/4 & 6 \end{bmatrix}$$

Pivoting: Practical Procedure III

- In the above procedure we use an array p to store the permutation
- We swap the whole two rows
- We can see that

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 18 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & -1/4 & 1 \end{bmatrix} \begin{bmatrix} 6 & 18 & -12 \\ 0 & 8 & 16 \\ 0 & 0 & 6 \end{bmatrix}$$

Pivoting: Practical Procedure IV

- Thus it is possible to get

$$PA = LU,$$

  where P is a permutation matrix

- We can overwrite A to get L and U, and use an array to store the permutation P

- To get x, we have

$$PA = Pb$$

  and then solve

$$(LU)x = Pb$$

Pivoting: Practical Procedure V

- The general algorithm:

```
for k=1:n-1
    find m
    swap A(k,1:n) and A(m, 1:n)
    p(k) = m;
    A(k+1:n,k) = A(k+1:n,k)/A(k,k);
    A(k+1:n, k+1:n) = A(k+1:n, k+1:n) -
        A(k+1:n,k)*A(k,k+1:n);
end
```

Pivoting: Practical Procedure VI

- General form:

$$M_{n-1}P_{n-1} \cdots M_1 P_1 A = U$$

Then

$$PA = LU \quad \text{and} \quad P = P_{n-1} \cdots P_1$$

- For the example,

$$P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P$$

Pivoting: Practical Procedure VII

- Why

$$PA = LU$$

  and L is the lower-triangular matrix obtained in the algorithm?

- By the definition of P, we hope

$$
\begin{aligned}
&PA\\
=&P_{n-1}\cdots P_1 A\\
=&P_{n-1}\cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1} U\\
=&LU
\end{aligned}
$$

Pivoting: Practical Procedure VIII

- Consider the ith column of

$$P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1}$$

- Note that for

$$P_j^{-1}, M_j^{-1}, j > i$$

the ith column is always

$$\begin{bmatrix} \vdots \\ 0 \\ \cdots \quad 1 \quad \cdots \\ 0 \\ \vdots \end{bmatrix}$$

Pivoting: Practical Procedure IX

- Thus

$$(P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1})_{:,i}$$
$$= (P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_i^{-1} M_i^{-1})_{:,i}$$

- Note that

$$(M_i^{-1})_{:,i} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ x \\ \vdots \\ x \end{bmatrix},$$

Pivoting: Practical Procedure X

so by a similar reason

$$(P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_i^{-1} M_i^{-1})_{:,i}$$
$$=(P_{n-1} \cdots P_{i+1} M_i^{-1})_{:,i} \tag{1}$$

- Here are details

$$(P_i^{-1} M_i^{-1})_{:,i} = \begin{bmatrix} \vdots \\ 0 \\ x' \\ x' \\ \vdots \\ x' \end{bmatrix},$$

because P_i swaps only two rows in $i, \ldots, n$. Then

$$(M_{i-1}^{-1} P_i^{-1} M_i^{-1})_{:,i}$$

$$= \begin{bmatrix} & 1 & & & \\ & x & 1 & & \\ \ldots & \vdots & 0 & 1 & \\ & \vdots & \vdots & & \ddots \\ & x & 0 & & \end{bmatrix} \begin{bmatrix} \vdots \\ 0 \\ x' \\ x' \\ \vdots \\ x' \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ x' \\ x' \\ \vdots \\ x' \end{bmatrix}$$

does not change the contents of $(P_i^{-1} M_i^{-1})_{:,i}$
Therefore,

$$\begin{aligned}
&(P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_i^{-1} M_i^{-1})_{:,i} \\
=&(P_{n-1} \cdots P_1 P_1^{-1} P_2^{-1} \cdots P_i^{-1} M_i^{-1})_{:,i} \\
=&(P_{n-1} \cdots P_{i+1} M_i^{-1})_{:,i},
\end{aligned}$$

where the last equality comes from

$$P_i^{-1} = P_i$$

- But (1) is exactly what we did in the algorithm: after M_i is obtained, we keep swapping some of its elements

$$(M_i)_{i+1:n,i}$$

by using

$$P_{n-1}, \cdots, P_{i+1}$$

- Example: consider

$$P_2 P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1}$$

in our example. For

$$M_1^{-1} P_2^{-1} M_2^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{bmatrix},$$

we really see that the first column is still

$$\begin{bmatrix} 1 \\ 1/3 \\ 1/2 \end{bmatrix}$$

- With

$$P_1^{-1} = P_1$$

in the end we have

$$P_2(M_1^{-1})_{:,1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1/3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix},$$

which is the first column of our L in (1)