Pivoting: Avoid Small Pivots I

- Pivoting in LU:

$$A = \begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 18 & -12 \end{bmatrix}$$

Largest absolute value in the first column: 6

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Pivoting: Avoid Small Pivots II

Then

$$P_1 A = \begin{bmatrix} 6 & 18 & -12 \\ 2 & 4 & -2 \\ 3 & 17 & 10 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$M_1 P_1 A = \begin{bmatrix} 6 & 18 & -12 \\ 0 & -2 & 2 \\ 0 & 8 & 16 \end{bmatrix}$$

Pivoting: Avoid Small Pivots III

- Swap 2nd and 3rd rows:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/4 & 1 \end{bmatrix}$$

$$M_2 P_2 M_1 P_1 A = \begin{bmatrix} 6 & 18 & -12 \\ 0 & 8 & 16 \\ 0 & 0 & 6 \end{bmatrix}$$

- The general form

$$M_{n-1} P_{n-1} \cdots M_1 P_1 A = U$$

Pivoting: Avoid Small Pivots IV

- Now we have

$$
\begin{aligned}
Ax &= b \\
(M_{n-1}P_{n-1}\cdots M_1P_1)Ax &= (M_{n-1}P_{n-1}\cdots M_1P_1)b
\end{aligned}
$$

  so we can solve

$$
Ux = (M_{n-1}P_{n-1}\cdots M_1P_1)b
$$

- But how to store $M_1, P_1, \ldots, M_{n-1}, P_{n-1}$?
- So we don't have a practical procedure yet

Pivoting: Practical Procedure I

- Let's try the following procedure by overwriting A

$$\begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 18 & -12 \end{bmatrix} \qquad p(1) = 3$$

$$\begin{bmatrix} 6 & 18 & -12 \\ 2 & 4 & -2 \\ 3 & 17 & 10 \end{bmatrix} \qquad \text{swap two rows}$$

Pivoting: Practical Procedure II

- Then

$$\begin{bmatrix} 6 & 18 & -12 \\ 1/3 & -2 & 2 \\ 1/2 & 8 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 18 & -12 \\ 1/2 & 8 & 16 \\ 1/3 & -2 & 2 \end{bmatrix} \qquad p(2) = 3$$

$$\begin{bmatrix} 6 & 18 & -12 \\ 1/2 & 8 & 16 \\ 1/3 & -1/4 & 6 \end{bmatrix}$$

Pivoting: Practical Procedure III

- In the above procedure we use an array p to store the permutation
- We swap the whole two rows
- We can see that

$$
\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 18 & -12 \end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & -1/4 & 1 \end{bmatrix}
\begin{bmatrix} 6 & 18 & -12 \\ 0 & 8 & 16 \\ 0 & 0 & 6 \end{bmatrix}
$$

Pivoting: Practical Procedure IV

- Thus it is possible to get

$$PA = LU,$$

where P is a permutation matrix

- We can overwrite A to get L and U, and use an array to store the permutation P
- To get x, we have

$$PA = Pb$$

and then solve

$$(LU)x = Pb$$

Pivoting: Practical Procedure V

- The general algorithm:

```
for k=1:n-1
    find m
    swap A(k,1:n) and A(m, 1:n)
    p(k) = m;
    A(k+1:n,k) = A(k+1:n,k)/A(k,k);
    A(k+1:n, k+1:n) = A(k+1:n, k+1:n) -
         A(k+1:n,k)*A(k,k+1:n);
end
```

Pivoting: Practical Procedure VI

- General form:

$$M_{n-1}P_{n-1}\cdots M_1P_1A = U$$

Then

$$PA = LU \quad \text{and} \quad P = P_{n-1}\cdots P_1$$

- For the example,

$$P_2P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P$$

Pivoting: Practical Procedure VII

- Why
$$PA = LU$$
and L is the lower-triangular matrix obtained in the algorithm?
- By the definition of P, we hope
$$\begin{aligned}
&PA\\
=&P_{n-1}\cdots P_1 A\\
=&P_{n-1}\cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1} U\\
=&LU
\end{aligned}$$

Pivoting: Practical Procedure VIII

- Consider the ith column of

$$P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1}$$

- Note that for

$$P_j^{-1}, M_j^{-1}, j > i$$

  the ith column is always

$$\begin{bmatrix} & \vdots & \\ & 0 & \\ \cdots & 1 & \cdots \\ & 0 & \\ & \vdots & \end{bmatrix}$$

Pivoting: Practical Procedure IX

- Thus

$$
\begin{aligned}
&(P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_{n-1}^{-1} M_{n-1}^{-1})_{:,i} \\
=&(P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_i^{-1} M_i^{-1})_{:,i}
\end{aligned}
$$

- Note that

$$
(M_i^{-1})_{:,i} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ x \\ \vdots \\ x \end{bmatrix},
$$

Pivoting: Practical Procedure X

so by a similar reason

$$
\begin{aligned}
&(P_{n-1} \cdots P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} \cdots P_i^{-1} M_i^{-1})_{:,i} \\
=&(P_{n-1} \cdots P_{i+1} M_i^{-1})_{:,i} \qquad (1)
\end{aligned}
$$

- But (1) is exactly what we did in the algorithm: after M_i is obtained, we keep swapping some of its elements

$$(M_i)_{i+1:n,i}$$

by using

$$P_{n-1}, \cdots, P_{i+1}$$

Pivoting: Practical Procedure XI

- Example: consider

$$P_2 P_1 P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1}$$

in our example. For

$$M_1^{-1} P_2^{-1} M_2^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{bmatrix},$$

Pivoting: Practical Procedure XII

we really see that the first column is still

$$\begin{bmatrix} 1 \\ 1/3 \\ 1/2 \end{bmatrix}$$

- With

$$P_1^{-1} = P_1$$

in the end we have

$$\begin{aligned} & P_2(M_1^{-1})_{:,1} \\ = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1/3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}, \end{aligned}$$

Pivoting: Practical Procedure XIII

which is the first column of our L in (1)