

Problems of LU Factorization I

- If the matrix A has zero diagonal elements such as

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then we cannot generate M_1

- Even if we can do LU factorization, numerical error is sometimes large
- Example 3.3.1 of Golub and Van Loan:
Assume $\beta = 10$, $p = 3$, **no guard digit**

Problems of LU Factorization II

Solve

$$\begin{bmatrix} 0.001 & 1.00 \\ 1.00 & 2.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 3.00 \end{bmatrix}$$

LU factorization

$$L = \begin{bmatrix} 1 & 0 \\ 1000 & 1 \end{bmatrix}, U = \begin{bmatrix} 0.001 & 1 \\ 0 & -1000 \end{bmatrix}$$

$$2 - 1000 \Rightarrow -1000$$

Problems of LU Factorization III

- The real solution

$$\begin{bmatrix} 0.001 & 1 \\ 0 & -998 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -997 \end{bmatrix}$$

$$x_2 = 997/998$$

$$x_1 = 1000(1 - 997/998) = 1000/998$$

Using our system:

$$\begin{bmatrix} 1 & 0 \\ 1000 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 3.00 \end{bmatrix}$$

Problems of LU Factorization IV

$$y_1 = 1.00, y_2 = 3 - 1000 = -1000$$

$$\begin{bmatrix} 0.001 & 1 \\ 0 & -1000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.00 \\ -1000 \end{bmatrix} \quad (1)$$

$$x_1 = 0, x_2 = 1$$

- $(0, 1)$ very different from $(1000/998, 997/998)$
- Conclusion:
A small pivot may cause large numerical errors
- Think about (1). If x_2 contains errors, because $1 \gg 0.001$, x_1 is seriously affected.

Problems of LU Factorization V

- If interchanging rows:

$$\begin{bmatrix} 1.00 & 2.00 \\ 0.001 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.001 & 1.00 \\ 1.00 & 2.00 \end{bmatrix} \Rightarrow \begin{bmatrix} 1.00 & 2.00 \\ 0.001 & 1.00 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 0.001 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Note that

$$1 - 2 \times 0.001 = 1$$

Problems of LU Factorization VI

$$\begin{bmatrix} 1 & 0 \\ 0.001 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$y_1 = 3, y_2 = 1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$x_1 = 1, x_2 = 1$$

(1, 1) much closer to (1000/998, 997/998)

Pivoting: Avoid Small Pivots I

- How to interchange rows:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0.001 & 1 \\ 1 & 2 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 2 \\ 0.001 & 1 \end{bmatrix}$$

Pivoting: Avoid Small Pivots II

- Permutation matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$4 \rightarrow 1, 1 \rightarrow 2, 3 \rightarrow 3, 2 \rightarrow 4$$