Interpolation and Approximation I

- If $f^{(n)}(x), \forall n$ are available, Taylor polynomial is an approximation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots$$

- Example:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- Need only information of one point

Interpolation and Approximation II

- If finite terms are selected, the approximation is accurate only near x_0
- Primal use of Taylor polynomials for numerical methods: derivation of numerical techniques but not approximation
- Subsequently we will discuss some interpolation and approximation techniques.
- Roughly speaking, we have:
    - Interpolation: find a function passing all the given points

Interpolation and Approximation III

- Approximation: find a function so error to the given points is minimized

Lagrange Polynomials I

- Given $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$, find a polynomial function passing all of them
- Why consider polynomials: the simplest form of functions
- Degree 1 polynomial passing two points: a straight line

Lagrange Polynomials II

- Given $(x_0, f(x_0)), (x_1, f(x_1))$. Define

$$L_0(x) \equiv \frac{x - x_1}{x_0 - x_1}, L_1(x) \equiv \frac{x - x_0}{x_1 - x_0},$$
$$P(x) \equiv L_0(x)f(x_0) + L_1(x)f(x_1)$$

  Then

$$P(x_0) = 1f(x_0) + 0f(x_1) = f(x_0)$$
$$P(x_1) = 0f(x_1) + 1f(x_1) = f(x_1)$$

- Generalization: higher-degree polynomials

Lagrange Polynomials III

- $(n+1)$ points $\Rightarrow$ consider a polynomial with degree at most n

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$$

- Construct $L_{n,k}(x)$. We hope that

$$L_{n,k}(x_i) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

Lagrange Polynomials IV

n: degree, k: index. Then

$$P(x) = \sum_{k=0}^{n} L_{n,k}(x) f(x_k)$$

Thus we have

$$P(x_i) = f(x_i)$$

- Define

$$L_{n,k}(x) \equiv \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

Lagrange Polynomials V

- Note that here we assume $x_i \neq x_j$

Spline Interpolation I

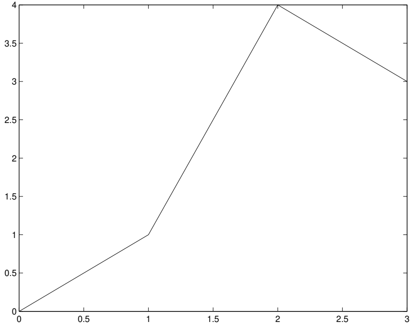
- Disadvantages of using Lagrange polynomials
  - Very high degrees
  - Large fluctuation
- We will instead try piecewise polynomial approximation
- Different polynomials on each interval
- Linear: a series of straight lines joining

$$(x_0, f(x_0)), \ldots, (x_n, f(x_n))$$

Spline Interpolation II

- Example

$$x = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}, y = \begin{bmatrix} 0 & 1 & 4 & 3 \end{bmatrix}$$



Spline Interpolation III

- Function forms: $s_0(x), \ldots, s_{n-1}(x)$

$$s_j(x) = \frac{x - x_{j+1}}{x_j - x_{j+1}} y_j + \frac{x - x_j}{x_{j+1} - x_j} y_{j+1}, x_j \le x \le x_{j+1}$$

- Disadvantage: not differentiable at end points

Spline Interpolation IV

- A possibility is to use higher-degree $s_j(x)$ and hope that

$$s'_0(x), \ldots, s'_{n-1}(x)$$

are continuous

- Thus we require that these functions satisfy

$$\begin{aligned} s_j(x_j) &= f(x_j), j = 0, \ldots, n-1, \\ s_{n-1}(x_n) &= f(x_n) \\ s_{j+1}(x_{j+1}) &= s_j(x_{j+1}), j = 0, \ldots, n-2 \\ s'_{j+1}(x_{j+1}) &= s'_j(x_{j+1}), j = 0, \ldots, n-2 \end{aligned}$$

Spline Interpolation V

- How many conditions:

$$n + 1 + 2(n - 1) = 3n - 1$$

- Quadratic piecewise interpolation:

$$3n \text{ variables}$$

- Each interval: a quadratic polynomial, 3 variables
- Most common piecewise approximation: cubic polynomials (Spline)

$$4n \text{ variables}$$