Discrete Analog to Fourier Series I

- Now we have

$$
\begin{aligned}
&\frac{\partial E}{\partial a_k} \\
= \;& 2\sum_{j=0}^{2m-1} y_j \cos kx_j - 2a_k \sum_{j=0}^{2m-1} \cos kx_j \cos kx_j \\
= \;& 0,
\end{aligned}
$$

so we must calculate

$$\sum_{j=0}^{2m-1} \cos kx_j \cos kx_j$$

Discrete Analog to Fourier Series II

This needs the following theorem

Theorem

If r isn't a multiple of m, then

$$\sum_{j=0}^{2m-1} (\cos rx_j)^2 = m \text{ and } \sum_{j=0}^{2m-1} (\sin rx_j)^2 = m$$

If

$$1 \leq k \leq n < m$$

we have

Discrete Analog to Fourier Series III

k isn't a multiple of m

By the theorem,

$$\sum_{j=0}^{2m-1} \cos kx_j \cos kx_j = m.$$

Finally

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j, k = 1, \ldots, n$$

Discrete Analog to Fourier Series IV

- We now need to handle the case of $k = 0$

Discrete Analog to Fourier Series V

$$
\begin{aligned}
-\frac{\partial E}{\partial a_0} &= 2\sum_{j=0}^{2m-1}\Big(y_j - \frac{a_0}{2} - \sum_{r=1}^{n} a_r \cos rx_j - \\
&\quad - \sum_{r=1}^{n-1} b_r \sin rx_j\Big)\frac{1}{2} \\
&= 2\sum_{j=0}^{2m-1}\left(y_j - \frac{a_0}{2}\right)\frac{1}{2} - \sum_{r=1}^{n} a_r \sum_{j=0}^{2m-1} \cos rx_j - \\
&\quad \sum_{r=1}^{n-1} b_r \sum_{j=0}^{2m-1} \sin rx_j \qquad (1) \\
&= 2\sum_{j=0}^{2m-1}\left(y_j - \frac{a_0}{2}\right)\frac{1}{2} = 0
\end{aligned}
$$

Discrete Analog to Fourier Series VI

- Note that in (1) we use the result from a theorem in the previous set of slides
- Now we have

$$\sum_{j=0}^{2m-1} y_j - m a_0 = 0$$

Because

$$\cos kx_j = 1 \text{ if } k = 0,$$

Discrete Analog to Fourier Series VII

we can write a_0 as

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j, k = 0$$

- This has the same form as when $k = 1, \ldots, n$
- This also explains why we use $a_0/2$ rather than a_0
- Similarly,

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j, k = 1, \ldots, n-1$$

Discrete Analog to Fourier Series VIII

- Why in the derivation we require $n < m$?
- Let's see what happens if $n = m$?
- When $k = n = m$,

$$\frac{\partial E}{\partial a_m} = 2\sum_{j=0}^{2m-1} y_j \cos mx_j - 2a_m \sum_{j=0}^{2m-1} \cos^2 mx_j = 0$$

Note that

$$x_j = -\pi + (\frac{j}{m})\pi, j = 0, \ldots, 2m-1$$

Discrete Analog to Fourier Series IX

Thus

$$(\cos mx_j)^2 = (\cos(j-m)\pi)^2 = 1, \forall j = 0, \ldots, 2m-1$$

Then

$$a_m = \frac{1}{2m} \sum_{j=0}^{2m-1} y_j \cos mx_j,$$

is different from the general form. It has $1/(2m)$ instead of $1/m$.

Discrete Analog to Fourier Series X

- To get the same form, $S_n(x)$ should be changed to

$$S_n(x) = \frac{a_0 + a_n \cos nx}{2} + \sum_{k=1}^{n-1} a_k \cos kx + \sum_{k=1}^{n-1} b_k \sin kx$$

- Therefore $n = m$ is also fine though earlier we used $n < m$ in our derivation.
- Interestingly, for FFT discussed later, we will consider $n = m$

Example I

- Let

$$f(x) = x^4 - 3x^3 + 2x^2 - \tan(x(x-2))$$

- Given $m = 5 \Rightarrow 10$ points

$$(x_j, y_j), j = 0, \ldots, 9. \qquad x_j = j/5, y_j = f(x_j)$$

- Need to linearly transform these points to $[-\pi, \pi]$

$$z_j = \pi(x_j - 1)$$

Example II

- Data become

$$z_j, f(1 + \frac{z_j}{\pi}), j = 0, \ldots, 9$$

- Let $n = 3$
- Least square trigonometric polynomial

$$S_3(z) = \frac{a_0}{2} + a_3 \cos 3z + \sum_{k=1}^{2} (a_k \cos kz + b_k \sin kz)$$

Example III

where

$$a_k = \frac{1}{5}\sum_{j=0}^{9} f(1+\frac{z_j}{\pi})\cos kz_j, k = 0, \ldots, 3$$

and

$$b_k = \frac{1}{5}\sum_{j=0}^{9} f(1+\frac{z_j}{\pi})\sin kz_j, k = 1, \ldots, 2$$

- Using Matlab

Example IV

```
m = 5
x=[0:1:2*m-1]'/5;
y = x.^4 - 3*x.^3 + 2*x.^2  - tan(x.*(x-2));
z = pi*(x-1);
n=3;
a = []; b = [];
for k=0:n
    a = [a ; 1/m*sum(y.*cos(k*z))];
end
for k=1:n-1
    b = [b; 1/m*sum(y.*sin(k*z))] ;
```

Example V

```
end
approx=a(1)/2 + cos(z*(1:n))*a(2:n+1) + ...
    sin(z*(1:n-1))*b(1:n-1)
% z*(1:n) is a matrix
>> [y approx]

ans =

         0    0.0011
    0.4340    0.4178
    0.8981    0.9107
```

Example VI

```
 1.3172    1.3208
 1.5820    1.5716
 1.5574    1.5578
 1.1980    1.2079
 0.6452    0.6414
 0.1301    0.1188
-0.1420   -0.1278
```

- The figure

Example VII