Discrete Analog to Fourier Series I

- Calculating integration is difficult!!
- Thus we don't consider the whole function. Instead we consider a discrete set of points
- Given

$$\{x_j, y_j\}, j = 0, \ldots, 2m-1$$

Consider $-\pi \leq x \leq \pi$

$$x_j = -\pi + (\frac{j}{m})\pi, j = 0, \ldots, 2m-1$$

Discrete Analog to Fourier Series II

- Assume $n < m$. Consider

$$\{\phi_0, \ldots, \phi_{2n-1}\}$$

such that

$$\phi_0(x) = \frac{1}{2}$$
$$\phi_k(x) = \cos kx, k = 1, \ldots, n$$
$$\phi_{n+k}(x) = \sin kx, k = 1, \ldots, n-1$$

Discrete Analog to Fourier Series III

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + \sum_{k=1}^{n-1} b_k \sin kx \\ &= \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) \end{aligned}$$

- Choose $a_0, \ldots, a_n, b_1, \ldots, b_{n-1}$ to minimize

$$E = \sum_{j=0}^{2m-1} (y_j - S_n(x_j))^2$$

Discrete Analog to Fourier Series IV

- Minimize

$$\sum_{j=0}^{2m-1} \left(y_j - \left(\frac{a_0}{2} + a_n \cos nx_j + \sum_{r=1}^{n-1} (a_r \cos rx_j + b_r \sin rx_j) \right) \right)^2$$

- First consider

$$1 \leq k \leq n$$

  and calculate

Discrete Analog to Fourier Series V

$$
\begin{aligned}
-\frac{\partial E}{\partial a_k} &= \sum_{j=0}^{2m-1} 2\Bigg(y_j - (\frac{a_0}{2} + a_n \cos nx_j + \\
&\quad \sum_{r=1}^{n-1}(a_r \cos rx_j + b_r \sin rx_j))\Bigg) \cos kx_j \\
&= 2\sum_{j=0}^{2m-1} y_j \cos kx_j - 2a_k \sum_{j=0}^{2m-1} \cos kx_j \cos kx_j \\
&\quad -2\sum_{r=1,r\neq k}^{n} \sum_{j=0}^{2m-1} a_r \cos rx_j \cos kx_j - \\
&\quad \sum_{j=0}^{2m-1} a_0 \cos kx_j - \\
&\quad 2\sum_{r=1}^{n-1} \sum_{j=0}^{2m-1} b_r \sin rx_j \cos kx_j
\end{aligned}
$$

Discrete Analog to Fourier Series VI

- If

$$\sum_{j=0}^{2m-1} \cos rx_j \cos kx_j = 0 \text{ if } r \neq k, \tag{1}$$

$$\sum_{j=0}^{2m-1} \sin rx_j \cos kx_j = 0, 1 \leq r \leq n-1, 1 \leq k \leq n, \tag{2}$$

and

$$\sum_{j=0}^{2m-1} \cos kx_j = 0, \tag{3}$$

Discrete Analog to Fourier Series VII

then

$$\begin{aligned} & \frac{\partial E}{\partial a_k} \\ = \ & 2\sum_{j=0}^{2m-1} y_j \cos kx_j - 2a_k \sum_{j=0}^{2m-1} \cos kx_j \cos kx_j = 0 \end{aligned}$$

- Similar to the continuous case, here we hope

  $$\phi_0, \ldots, \phi_{2n-1}$$

  are orthogonal with respect to summation over equally spaced points $x_0, \ldots, x_{2m-1}$ in $[-\pi, \pi]$.

Discrete Analog to Fourier Series VIII

- Eqs. (1)-(3) are part of the above property
- To prove (1)-(3), we need the following theorem:

Theorem

If r isn't a multiple of $2m$, then

$$\sum_{j=0}^{2m-1} \cos rx_j = 0 \text{ and } \sum_{j=0}^{2m-1} \sin rx_j = 0$$

- Proofs are omitted.

Discrete Analog to Fourier Series IX

- Proof of (1). From

$$\cos rx_j \cos kx_j = \frac{1}{2}(\cos(r+k)x_j + \cos(r-k)x_j)$$

Now

$$1 \leq r \leq n, 1 \leq k \leq n, r \neq k.$$

Then

$$1 \leq r+k < 2n < 2m$$

so

$$r+k \text{ isn't a multiple of } 2m.$$

Discrete Analog to Fourier Series X

From the theorem,

$$\sum_{j=0}^{2m-1} \cos(r+k)x_j = 0$$

For $r-k$,

$$-2m < -n \leq r-k \neq 0 \leq n < 2m$$

so

$$r-k \text{ isn't a multiple of } 2m.$$

Discrete Analog to Fourier Series XI

From the theorem,

$$\sum_{j=0}^{2m-1} \cos(r-k)x_j = 0$$

Proofs of (2) and (3) are similar