Trigonometric Polynomial Approximation I

- Consider $\{\phi_0, \ldots, \phi_{2n-1}\}$ such that

$$\phi_0(x) = \frac{1}{2}$$
$$\phi_k(x) = \cos kx, k = 1, \ldots, n$$
$$\phi_{n+k}(x) = \sin kx, k = 1, \ldots, n-1$$

- Some derivations include $\sin nx$ as well, though we don't do that here
- These functions are orthogonal on $[-\pi, \pi]$:

Trigonometric Polynomial Approximation II

If $k \neq j, k \in \{1, \ldots, n-1\}, j \in \{1, \ldots, n\}$

$$\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_j(x)dx = \int_{-\pi}^{\pi} \sin kx \cos jx dx$$

- Using

$$\sin kx \cos jx = \frac{1}{2}\sin(k+j)x + \frac{1}{2}\sin(k-j)x$$

Trigonometric Polynomial Approximation III

$$\begin{aligned}
&\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_j(x)dx \\
= \;& \frac{1}{2}\int_{-\pi}^{\pi}(\sin(k+j)x + \sin(k-j)x)dx \\
= \;& \frac{1}{2}\left(\frac{-\cos(k+j)x}{k+j} - \frac{-\cos(k-j)x}{k-j}\right)\Bigg|_{-\pi}^{\pi} = 0
\end{aligned}$$

Note that here we used the property

$$\cos(x) = \cos(-x)$$

Trigonometric Polynomial Approximation IV

- What if $k = j$?

$$\begin{aligned}\int_{-\pi}^{\pi} \sin kx \cos kx dx =& \frac{1}{2}\int_{-\pi}^{\pi} \sin 2kx dx \\ =& \frac{-1}{4k} \cos 2kx \Big|_{-\pi}^{\pi} = 0\end{aligned}$$

This case is singled out because of $k - j = 0$ in the denominator of the previous equation

Trigonometric Polynomial Approximation V

- There are two other cases:

$$k, j, k = 1, \ldots, n, j = 1, \ldots, n$$
$$n + k, n + j, k = 1, \ldots, n - 1, j = 1, \ldots, n - 1$$

  Also we need to check $k = 0$

- They are also orthogonal though details are omitted here

Orthogonal Trigonometric Polynomials I

- Let

$$S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx)$$

  We will see why $a_0/2$ rather than a_0 is used

- With the orthogonality,

Orthogonal Trigonometric Polynomials II

$$a_k = \frac{\int_{-\pi}^{\pi} f(x)\cos kx dx}{\int_{-\pi}^{\pi} (\cos kx)^2 dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx, \quad (1)$$

$$k = 1, \ldots, n$$

$$b_k = \frac{\int_{-\pi}^{\pi} f(x)\sin kx dx}{\int_{-\pi}^{\pi} (\sin kx)^2 dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx,$$

$$k = 1, \ldots, n-1$$

Orthogonal Trigonometric Polynomials III

- For the denominator, we can easily calculate

$$\int_{-\pi}^{\pi} (\cos kx)^2 dx = \int_{-\pi}^{\pi} (\sin kx)^2 dx = \pi$$

  though details are not shown
- Why do we use

$$\frac{a_0}{2}$$

  rather than

$$a_0$$

Orthogonal Trigonometric Polynomials IV

Reason:

$$a_0 = \frac{\int_{-\pi}^{\pi} f(x)\frac{1}{2}dx}{\int_{-\pi}^{\pi} \frac{1}{4}dx}$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 0x dx$$

Then a_0 has the same form as (1)

- Q: why considering $\cos kx$ as well as $\sin kx$? Why not $\cos kx$ only?
- For the convergence theory of Fourier series (see below), we need both $\cos kx$ and $\sin kx$.

Fourier Series I

- The function we have considered is

$$S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx)$$

- When $n \to \infty$,

$$\lim_{n\to\infty} S_n(x)$$

  is called the "Fourier series" of $f(x)$
- Under certain condition of $f(x)$,

$$S_n(x) \to f(x), \forall x \text{ as } n \to \infty$$

Example of Fourier Series I

- Consider

$$f(x) = |x|, \forall -\pi < x < \pi$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{2}{\pi}\int_0^{\pi} x dx = \pi$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos kx dx = \frac{2}{\pi}\int_0^{\pi} x \cos kx dx \qquad (2)$$

Example of Fourier Series II

$$\begin{aligned}\int_0^\pi x \cos kx dx &= x\frac{\sin kx}{k}\Big|_0^\pi - \int_0^\pi \frac{\sin kx}{k} dx \\ &= \frac{\pi \sin k\pi}{k} + \frac{\cos kx}{k^2}\Big|_0^\pi \\ &= 0 + \frac{(-1)^k - 1}{k^2} \\ &= \frac{(-1)^k - 1}{k^2}\end{aligned}$$

Therefore, from (2),

$$a_k = \frac{2}{\pi k^2}((-1)^k - 1), \forall k = 1, \ldots, n$$

Example of Fourier Series III

- Because

$$\sin x = -\sin(-x)$$

we have

$$|x| \sin x = -|-x| \sin(-x)$$

and therefore

$$b_k = \int_{-\pi}^{\pi} |x| \sin kx dx = 0$$

Example of Fourier Series IV

- Finally

$$S_n(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos kx$$

- We can see

$$S_0(x) = \frac{\pi}{2}, \text{ a straight line}$$
$$S_1(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x$$
$$S_2(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x + 0$$
$$S_3(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x - \frac{4}{9\pi} \cos 3x$$

Example of Fourier Series V

- This is an example where we use $\cos kx$ only