# Continuous Least Square I

- Reference: Section 8 of Richard L. Burden and J. Douglas Faires, Numerical Analysis, 9th Edition.
- Given $f \in C[a, b]$, i.e., continuous functions on $[a, b]$
  Find a polynomial

$$p_n(x) = a_n x^n + \cdots + a_0 = \sum_{k=0}^{n} a_k x^k$$

# Continuous Least Square II

to minimize

$$
\begin{aligned}
E &= \int_a^b (f(x) - p_n(x))^2 dx \\
&= \int_a^b (f(x) - \sum_{k=0}^{n} a_k x^k)^2 dx
\end{aligned}
$$

- Let

$$\frac{\partial E}{\partial a_j} = 0, j = 0, \ldots, n$$

# Continuous Least Square III

- Now

$$E = \int_a^b f(x)^2 dx - 2\sum_{k=0}^{n} \int_a^b a_k x^k f(x) dx + \int_a^b (\sum_{k=0}^{n} a_k x^k)^2 dx$$

So

$$\frac{\partial E}{\partial a_j} = -2\int_a^b x^j f(x) dx + \int_a^b 2(\sum_{k=0}^{n} a_k x^k) x^j dx$$

$$= -2\int x^j f(x) dx + 2\sum_{k} a_k \int x^{k+j} dx = 0$$

# Continuous Least Square IV

- There are $(n+1)$ linear equations

$$\sum_{k=0}^{n} a_k \int_a^b x^{k+j} dx = \int_a^b x^j f(x) dx, j = 0, \ldots, n$$

# Example I

- $f(x) = \sin \pi x$ on $[0, 1]$
- Approximation by a polynomial of degree 2

$$a_0 \int_0^1 1dx + a_1 \int_0^1 xdx + a_2 \int_0^1 x^2 dx = \int_0^1 \sin \pi x dx$$

$$a_0 \int_0^1 xdx + a_1 \int_0^1 x^2 dx + a_2 \int_0^1 x^3 dx = \int_0^1 x \sin \pi x dx$$

$$a_0 \int_0^1 x^2 dx + a_1 \int_0^1 x^3 dx + a_2 \int_0^1 x^4 dx = \int_0^1 x^2 \sin \pi x dx$$

# Example II

$$\begin{aligned}\int_0^1 \sin \pi x dx &= \frac{-1}{\pi} \cos \pi x \Big|_0^1 \\ &= \frac{-1}{\pi}(-1-1) = \frac{2}{\pi}\end{aligned}$$

# Example III

$$
\begin{aligned}
\int_0^1 x \sin \pi x dx &= \frac{-1}{\pi} x \cos \pi x \Big|_0^1 - \int_0^1 \frac{-1}{\pi} \cos \pi x dx \\
&= \frac{1}{\pi} + \frac{1}{\pi}\frac{1}{\pi} \sin \pi x \Big|_0^1 \\
&= \frac{1}{\pi}
\end{aligned}
$$

# Example IV

$$
\begin{aligned}
& \int_0^1 x^2 \sin \pi x dx \\
= \; & \frac{-1}{\pi} x^2 \cos \pi x \Big|_0^1 - \frac{-1}{\pi} \int_0^1 2x \cos \pi x dx \\
= \; & \frac{1}{\pi} + \frac{2}{\pi} \Big( \frac{1}{\pi} x \sin \pi x \Big|_0^1 - \int_0^1 \frac{1}{\pi} \sin \pi x dx \Big) \\
= \; & \frac{1}{\pi} + \frac{2}{\pi} \Big( 0 + \frac{1}{\pi^2} \cos \pi x \Big|_0^1 \Big) \\
= \; & \frac{1}{\pi} + \frac{2}{\pi} \Big( \frac{-2}{\pi^2} \Big) = \frac{1}{\pi} - \frac{4}{\pi^3}
\end{aligned}
$$

# Example V

- Three equations:

$$a_0 = \frac{12\pi^2 - 120}{\pi^3} \approx -0.050465$$
$$a_1 = -a_2 = \frac{720 - 60\pi^2}{\pi^3} \approx 4.12251$$

- The approximate function

$$p_2(x) = -4.12251x^2 + 4.12251x - 0.050465$$

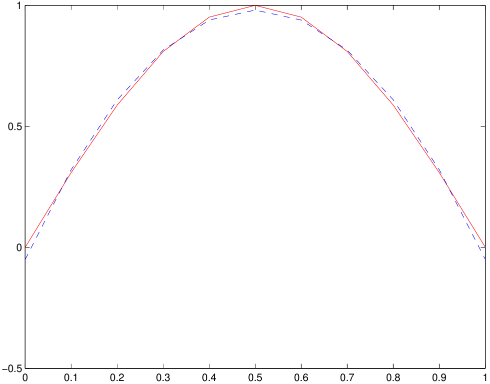
- A Matlab program

# Example VI

```
x = 0:0.1:1;
y = sin(pi*x);
p = [-4.12251 4.12251  -0.050465];
z = polyval(p,x);
plot(x,y,'r-', x, z, 'b--')
```

- The figure

# Example VII




- Difficulties:

  Need to solve an $(n+1)$ by $(n+1)$ linear system

# Linearly Independent Functions I

- Functions $\{\phi_0, \ldots, \phi_n\}$ are linearly independent on $[a, b]$ if, whenever

$$c_0\phi_0(x) + \cdots + c_n\phi_n(x) = 0, \forall x \in [a, b]$$

  we have

$$c_0 = \cdots = c_n = 0$$

- Linearly independent set of polynomials

  If

$$\phi_j \text{ is a polynomial of degree } j,$$

  then

# Linearly Independent Functions II

$\{\phi_0, \ldots, \phi_n\}$ are linearly independent on any $[a, b]$

- Proof omitted
- Any polynomial with degree $\leq n$, can be written uniquely as a linear combination of such $\{\phi_0, \ldots, \phi_n\}$
- If $\{\phi_0, \ldots, \phi_n\}$ are linearly independent and

$$p_n(x) = \sum_{k=0}^{n} a_k \phi_k(x),$$

# Linearly Independent Functions III

then

$$\min E = \int_a^b (f(x) - \sum_{k=0}^{n} a_k \phi_k(x))^2 dx$$

leads to

$$0 = -\frac{\partial E}{\partial a_j} = 2 \int_a^b (f(x) - \sum_{k=0}^{n} a_k \phi_k(x)) \phi_j(x) dx$$

# Linearly Independent Functions IV

- For $j = 0, \ldots, n$

$$\sum_{k=0}^{n} a_k \int_a^b \phi_k(x)\phi_j(x)dx = \int_a^b f(x)\phi_j(x)dx \quad (1)$$

- If $\phi_0, \ldots, \phi_n$ are chosen so that

$$\int_a^b \phi_k(x)\phi_j(x)dx = \begin{cases} 0 & \text{if } j \neq k \\ \alpha_k > 0 & \text{if } j = k \end{cases}$$

we say they are orthogonal

# Linearly Independent Functions V

- If $\alpha_k = 1, \forall k$, they are orthonormal
- Why $\alpha_k > 0$

$$\int_a^b \phi_k(x)^2 dx \text{ should be positive}$$

- (1) becomes

$$a_j \alpha_j = \int_a^b f(x) \phi_j(x) dx$$

# Linearly Independent Functions VI

- That is

$$a_j = \frac{1}{\alpha_j} \int_a^b f(x)\phi_j(x)dx$$