# Fast Fourier Transformation I

- Consider $n = m$

$$S_m(x) = \frac{a_0 + a_m \cos mx}{2} + \sum_{k=1}^{m-1} (a_k \cos kx + b_k \sin kx)$$

Then

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j, k = 0, \ldots, n \tag{1}$$

# Fast Fourier Transformation II

and

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j, k = 1, \ldots, n-1 \qquad (2)$$

- This calculation requires

  $$(2m)^2 \text{ multiplications, } (2m)^2 \text{ additions}$$

- FFT:

  $O(m \log_2 m)$ multiplications, $O(m \log_2 m)$ additions
- FFT is named by SIAM as one of the top 10 algorithms in the 20th century

# Fast Fourier Transformation III

- Extend $b_k$'s range from $1, \ldots, n-1$ to $0, \ldots, n$

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j, k = 0, \ldots, m$$

as

$$\sin 0x_j = 0 \text{ and } \sin mx_j = \sin(m(-\pi + \frac{\pi j}{m})) = 0$$

imply

$$b_0 = 0 = b_m$$

# Fast Fourier Transformation IV

- Define

$$c_k = \sum_{j=0}^{2m-1} y_j e^{-\pi ijk/m}, k = 0, \ldots, 2m-1 \qquad (3)$$

- We have

# Fast Fourier Transformation V

$$
\begin{aligned}
c_k &= \sum_{j=0}^{2m-1} y_j e^{-\pi ijk/m} \\
&= \sum_{j=0}^{2m-1} y_j e^{ik(-\pi j/m+2\pi)} = \sum_{j=0}^{2m-1} y_j e^{ik(\pi-\pi j/m+\pi)} \\
&= \sum_{j=0}^{2m-1} y_j e^{-ikx_j} e^{ik\pi} = m(a_k - ib_k)(\cos k\pi + i \sin k\pi) \\
&= m(a_k - ib_k)(-1)^k,
\end{aligned}
$$

# Fast Fourier Transformation VI

where in one place we use

$$x_j = -\pi + \frac{\pi j}{m},$$

and $a_k$ and $b_k$ are from (1) and (2), respectively.

- If $c_k$ is known, then from the above derivation

$$\frac{c_k(-1)^k}{m} = (a_k - ib_k)(-1)^k(-1)^k = (a_k - ib_k)$$

# Fast Fourier Transformation VII

- The above equation also implies

$$a_k = \text{Re}\left(\frac{c_k(-1)^k}{m}\right)$$
$$b_k = -\text{Im}\left(\frac{c_k(-1)^k}{m}\right)$$

Re: real part, Im: imaginary part

# Recursive Form of FFT I

- $m = 2$

$$x_j = -\pi + \left(\frac{j}{2}\right)\pi, j = 0, \ldots, 3$$

$$S_2(x) = \frac{a_0 + a_2 \cos 2x}{2} + a_1 \cos x + b_1 \sin x$$

- Using (3)

$$\begin{array}{rcrcrcr} c_0 = y_0 e^0 & + & y_1 e^0 & + & y_2 e^0 & + & y_3 e^0, \\ c_1 = y_0 e^0 & + & y_1 e^{-\pi i/2} & + & y_2 e^{-\pi i} & + & y_3 e^{-3\pi i/2}, \\ c_2 = y_0 e^0 & + & y_1 e^{-\pi i} & + & y_2 e^{-2\pi i} & + & y_3 e^{-3\pi i}, \\ c_3 = y_0 e^0 & + & y_1 e^{-3\pi i/2} & + & y_2 e^{-3\pi i} & + & y_3 e^{-9\pi i/2} \end{array}$$

# Recursive Form of FFT II

- There are other ways to define $c_k$. For example, if

$$c_k = \sum_{j=0}^{2m-1} y_j e^{\pi ijk/m}, k = 0, \ldots, 2m-1$$

then

$$\begin{aligned} c_k &= \sum_{j=0}^{2m-1} y_j e^{ik(-\pi+\pi j/m-\pi)} = \sum_{j=0}^{2m-1} y_j e^{ikx_j} e^{-ik\pi} \\ &= m(a_k + ib_k)(\cos k\pi - i \sin k\pi) \\ &= m(a_k + ib_k)(-1)^k, \end{aligned}$$

# Recursive Form of FFT III

where $a_k$ and $b_k$ are from (1) and (2), respectively.

- We do not consider this way of defining $c_k$. Instead we use (3) by following the commonly used form of FFT
- Now we need a systematic way to calculate

$$c_k, k = 0, \ldots, 2m-1$$

and then find

$$a_k, b_k, k = 0, \ldots, m$$

# Recursive Form of FFT IV

- We have

$$\begin{bmatrix} \vdots \\ c_k \\ \vdots \end{bmatrix} = Fy,$$

where

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \delta & \delta^2 & \cdots & \delta^{(2m-1)} \\ 1 & \delta^2 & \delta^4 & \cdots & \delta^{2(2m-1)} \\ & & \vdots & & \\ 1 & \delta^{(2m-1)} & \delta^{2(2m-1)} & \cdots & \delta^{(2m-1)^2} \end{bmatrix}$$

# Recursive Form of FFT V

and

$$\delta = e^{-i\pi/m}$$

- Note that

$$e^{-i\pi/m} \text{ is the } (2m)\text{th root of unity}$$

because

$$(e^{-i\pi/m})^{2m} = e^{-i2\pi} = 1$$