Vector and Matrix Norms I

- We have different types of objects: scalar, vector, matrix

  How to calculate errors?
- Scalar:

  absolute error:

$$|\hat{\alpha} - \alpha|$$

  relative error:

$$\frac{|\hat{\alpha} - \alpha|}{|\alpha|}$$

- Vectors: vector norm

Vector and Matrix Norms II

Norm is a way to calculate the length of a vector

- l-norm:

$$\|x\|_l = \sqrt[l]{|x_1|^l + \cdots + |x_n|^l}$$

- 1-norm:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Vector and Matrix Norms III

- ∞-norm: it is defined as

$$\|x\|_\infty \equiv \lim_{l\to\infty} \|x\|_l$$

We can then do the following calculation

$$\lim_{l\to\infty} \|x\|_l = \lim_{l\to\infty} \sqrt[l]{|x_1|^l + \cdots + |x_n|^l}$$
$$= \max |x_i|$$

Proof:

$$\sqrt[l]{(\max|x_i|)^l} \le \sqrt[l]{|x_1|^l + \cdots + |x_n|^l} \le \sqrt[l]{n(\max|x_i|)^l}$$

Vector and Matrix Norms IV

- Example:

$$x = \begin{bmatrix} 1 \\ 100 \\ 9 \end{bmatrix} \text{ and } \hat{x} = \begin{bmatrix} 1.1 \\ 99 \\ 11 \end{bmatrix}$$

$$\|\hat{x} - x\|_\infty = 2, \frac{\|\hat{x}-x\|_\infty}{\|x\|_\infty} = 0.02, \frac{\|\hat{x}-x\|_\infty}{\|\hat{x}\|_\infty} = 0.0202$$

$$\|\hat{x} - x\|_2 = 2.238, \frac{\|\hat{x}-x\|_2}{\|x\|_2} = 0.0223, \frac{\|\hat{x}-x\|_2}{\|\hat{x}\|_2} = 0.0225$$

- For l-norm, we say that all norms are equivalent

Vector and Matrix Norms V

- For l_1 and l_2 norms, there exist c_1 and c_2 such that

$$c_1\|x\|_{l_1} \leq \|x\|_{l_2} \leq c_2\|x\|_{l_1}$$

  for all $x \in R^n$

- Example:

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \quad (1)$$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty \quad (2)$$

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty \quad (3)$$

  Proofs are omitted as they are easy

Vector and Matrix Norms VI

- Therefore, you can just choose a norm for your convenience
- Matrix norm: How to define the distance between two matrices ?
- Usually a norm should satisfy

$$
\begin{aligned}
\|A\| &\geq 0 \\
\|A + B\| &\leq \|A\| + \|B\| \\
\|\alpha A\| &= |\alpha| \|A\|,
\end{aligned} \tag{4}
$$

where α is a scalar

Vector and Matrix Norms VII

- Definition:

$$\|A\|_l \equiv \max_{x \neq 0} \frac{\|Ax\|_l}{\|x\|_l} = \max_{\|x\|_l = 1} \|Ax\|_l$$

- Proof of (4)

$$\|A + B\| = \max_{\|x\|=1} \|(A + B)x\|$$

$$\leq \max_{\|x\|=1} (\|Ax\| + \|Bx\|) \leq \max_{\|x\|=1} \|Ax\| + \max_{\|x\|=1} \|Bx\|$$

Vector and Matrix Norms VIII

- From this definition,

$$\frac{\|Ax\|}{\|x\|} \leq \|A\|, \forall x$$

so

$$\|Ax\| \leq \|A\|\|x\|$$

Relative error I

- Usually calculating

$$\frac{|\hat{\alpha} - \alpha|}{|\alpha|}$$

  is not practical because α is unknown
- A more reasonable estimate is

$$\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|}$$

Relative error II

- If

$$\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|} \le 0.1,$$

then

$$\frac{1}{1.1}\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|} \le \frac{|\hat{\alpha} - \alpha|}{|\alpha|} \le \frac{1}{0.9}\frac{|\hat{\alpha} - \alpha|}{|\hat{\alpha}|}$$

Proof:

$$|\alpha| - |\hat{\alpha}| \le |\hat{\alpha} - \alpha| \le 0.1|\hat{\alpha}|$$
$$|\alpha| \le 1.1|\hat{\alpha}|$$

Relative error III

Similarly,

$$|\hat{\alpha}| - |\alpha| \leq 0.1|\hat{\alpha}|$$
$$|\alpha| \geq 0.9|\hat{\alpha}|$$

Condition of a Linear System I

- Solving a linear system

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix}, \text{ solution } = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Right-hand side slightly modified

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{bmatrix}, \text{ solution } = \begin{bmatrix} 9.2 \\ -12.6 \\ 4.5 \\ -1.1 \end{bmatrix}$$

Condition of a Linear System II

A small modification causes a huge error

- Matrix slightly modified

$$\begin{bmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.89 & 9 \\ 6.99 & 4.99 & 9 & 9.98 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix}$$

$$\text{solution} = \begin{bmatrix} -81 \\ 137 \\ -34 \\ 22 \end{bmatrix}$$

Condition of a Linear System III

- Right-hand side slightly modified

$$
\begin{aligned}
Ax &= b \\
A(x + \delta x) &= b + \delta b \\
\delta x = A^{-1}\delta b &\Rightarrow \|\delta x\| \le \|A^{-1}\|\|\delta b\| \\
\|b\| &\le \|A\|\|x\| \\
\frac{\|\delta x\|}{\|x\|} &\le \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}
\end{aligned}
$$

Condition of a Linear System IV

- Matrix modified

$$
\begin{aligned}
Ax &= b \\
(A + \delta A)(x + \delta x) &= b \\
Ax + A\delta x + \delta A(x + \delta x) &= b \\
\delta x &= -A^{-1}\delta A(x + \delta x) \\
\|\delta x\| &\le \|A^{-1}\|\|\delta A\|\|x + \delta x\| \\
\frac{\|\delta x\|}{\|x + \delta x\|} &\le \|A\|\|A^{-1}\|\frac{\|\delta A\|}{\|A\|}
\end{aligned}
$$

Condition of a Linear System V

Note that here we can estimate only

$$\frac{\|\delta x\|}{\|x + \delta x\|} \text{ instead of } \frac{\|\delta x\|}{\|x\|};$$

see the discussion on relative errors earlier

- Clearly, error is strongly related to $\|A\|\|A^{-1}\|$, which is defined as the condition of A
- A smaller condition number is better