# Solving Dense Linear Systems

- Solving $Ax = b$ is an important numerical method
- We assume that $A$ is dense
- That is, most

  $$A_{ij} \neq 0$$

  and we allocate space to store

  $$A_{ij}, \forall i, \forall j$$

  Later we will discuss sparse matrices, where most $A_{ij}$ are zero.
- We start with triangular systems

# Solving Triangular Systems I

- Triangular system:

$$\begin{bmatrix} l_{11} & \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

if $l_{11}, l_{22} \neq 0$,

$$\begin{aligned} x_1 &= b_1/l_{11} \\ x_2 &= (b_2 - l_{21}x_1)/l_{22} \end{aligned}$$

In general

$$x_i = (b_i - \sum_{j=1}^{i-1} l_{ij}x_j)/l_{ii}$$

# Solving Triangular Systems II

- Require $O(n^2)$ operations
- This is "forward substitution"
- Back substitution for upper triangular system

$$x_i = (b_i - \sum_{j=i+1}^{n} u_{ij} x_j) / u_{ii}$$

# Solving Triangular Systems III

- Column oriented version

$$\begin{bmatrix} 2 & 0 & 0 \\ 6 & 5 & 0 \\ 7 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

$x_1 = 3$

$$\begin{bmatrix} 5 & 0 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3 \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} -16 \\ -16 \end{bmatrix}$$

# LU factorization I

- An example

$$
\begin{aligned}
3x_1 + 5x_2 &= 9 \\
6x_1 + 7x_2 &= 4
\end{aligned}
$$

By Gaussian elimination:

$$
\begin{array}{rcl}
3x_1 + 5x_2 & = & 9 \\
-3x_2 & = & -14
\end{array}
$$

Actually

$$
\begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & -3 \end{bmatrix}
$$

# LU factorization II

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -14 \end{bmatrix}$$

Then solve

$$\begin{bmatrix} 3 & 5 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -14 \end{bmatrix}$$

- Gauss transformation

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

# LU factorization III

- If

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

then

$$M_1 A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix}$$

- Note that

$$\begin{bmatrix} -3 & -6 \\ -6 & -11 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 6 & 10 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 7 \end{bmatrix}$$

# LU factorization IV

- Here 1 is called the pivot
- Likewise,

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow M_2(M_1 A) = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $-3$ is the pivot

# LU factorization V

Actually the whole Gaussian elimination is

$$U = M_{n-1} \cdots M_2 M_1 A$$

$U$: upper triangular

$$A = (M_1^{-1}(M_2^{-1} \cdots (M_{n-1}^{-1} U))) = LU$$

$M_1^{-1} \cdots M_{n-1}^{-1}$ are lower triangular matrices

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

# LU factorization VI

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} M_1^{-1} M_2^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \end{aligned}$$

# LU factorization VII

- We do not really store $M_1, \ldots, M_{n-1}$
- They can be stored in the lower-left part of $A$
- Why LU factorization but not Gaussian elimination ?
- Gaussian elimination overwrites $A$. If you do not overwrite $A$, need another array (the same size as $A$ and $L + U$)

# LU factorization VIII

- Gaussian elimination update right-hand side together

$$
\begin{aligned}
3x_1 + 5x_2 &= 9 \\
6x_1 + 7x_2 &= 4
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
3x_1 + 5x_2 &= 9 \\
-3x_2 &= -14
\end{aligned}
$$

# LU factorization IX

To handle multiple right-hand sides, LU is the most effective way

- Algorithm

```
for k=1:n-1
    A(k+1:n,k) = A(k+1:n,k)/A(k,k);
    A(k+1:n, k+1:n) = A(k+1:n, k+1:n) -
          A(k+1:n,k)*A(k,k+1:n);
end
```

# LU factorization X

- Number of operations

$$\approx \sum_{k=1}^{n} 2k^2 \approx 2\frac{n(n+1)(2n+1)}{6} \approx \frac{2n^3}{3}$$

Why $2k^2$: each element: one multiplication and one subtraction