Convergence Rate I

- To see which algorithm needs fewer iterations, a way for the analysis is the convergence rate
- Assume x^* is a solution
- An algorithm has linear convergence if

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \leq r < 1,$$

where r is a constant

Convergence Rate II

- Example: $r = 0.1$, and

$$x_1 - x^* = 0.1$$
$$x_2 - x^* = 0.01$$
$$x_3 - x^* = 0.001$$

- Superlinear convergence: if

$$\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0$$

Convergence Rate III

Example:

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0.1, 0.01, 0.001, \ldots$$

$$\begin{aligned} x_1 - x^* &= 0.1 \\ x_2 - x^* &= 0.1 \times 0.1 = 0.01 \\ x_3 - x^* &= 0.01 \times 0.01 = 0.0001 \\ x_4 - x^* &= 0.001 \times 0.0001 = 10^{-7} \end{aligned}$$

Convergence Rate IV

- Quadratic convergence. If

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} \le r \tag{1}$$

Example: $r = 0.1$

$$\begin{aligned}
x_1 - x^* &= 0.1 \\
x_2 - x^* &= (0.1)^2 \times 0.1 = 10^{-3} \\
x_3 - x^* &= (10^{-3})^2 \times 0.1 = 10^{-7} \\
x_4 - x^* &= (10^{-7})^2 \times 0.1 = 10^{-15}
\end{aligned}$$

Convergence Rate V

- No need to have $r < 1$
- As long as

$$\|x_k - x^*\| \to 0$$

we have that (1) implies superlinear convergence:

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \leq r \lim_{k\to\infty} \|x_k - x^*\| = 0$$

- Thus even with $r > 1$ the quadratic convergence is still faster than superlinear

Convergence Rate VI

- Example:

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} = 2^2, \text{ and start from } \|x_k - x^*\| = 2^{-3}$$

$$2^{-6} \cdot 2^2 = 2^{-4}, 2^{-8} \cdot 2^2 = 2^{-6}, \ldots$$

- We see that the convergence rates aim to see the situation when x_k is close to x^* (so sometimes we call it "local convergence rate")
- Newton method: quadratic convergence