To see which algorithm needs fewer iterations, a way for the analysis is the convergence rate.

Assume $x^*$ is a solution.

An algorithm has linear convergence if

$$\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \leq r < 1,$$

where $r$ is a constant.
Example: $r = 0.1$, and

\[
\begin{align*}
  x_1 - x^* &= 0.1 \\
  x_2 - x^* &= 0.01 \\
  x_3 - x^* &= 0.001
\end{align*}
\]

Superlinear convergence: if

\[
\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0
\]
Convergence Rate III

Example:

\[
\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0.1, 0.01, 0.001, \ldots
\]

\[
x_1 - x^* = 0.1
\]
\[
x_2 - x^* = 0.1 \times 0.1 = 0.01
\]
\[
x_3 - x^* = 0.01 \times 0.01 = 0.0001
\]
\[
x_4 - x^* = 0.001 \times 0.0001 = 10^{-7}
\]
Quadratic convergence. If

\[ \lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} \leq r \]  

(1)

Example: \( r = 0.1 \)

\[
\begin{align*}
x_1 - x^* &= 0.1 \\
x_2 - x^* &= (0.1)^2 \times 0.1 = 10^{-3} \\
x_3 - x^* &= (10^{-3})^2 \times 0.1 = 10^{-7} \\
x_4 - x^* &= (10^{-7})^2 \times 0.1 = 10^{-15}
\end{align*}
\]
Convergence Rate V

- No need to have $r < 1$
- As long as

$$\|x_k - x^*\| \to 0$$

we have that (1) implies superlinear convergence:

$$\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \leq r \lim_{k \to \infty} \|x_k - x^*\| = 0$$

- Thus even with $r > 1$ the quadratic convergence is still faster than superlinear
Example:

\[
\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} = 2^2, \text{ and start from } \|x_k - x^*\| = 2^{-3}
\]

\[2^{-6} \cdot 2^2 = 2^{-4}, 2^{-8} \cdot 2^2 = 2^{-6}, \ldots\]

We see that the convergence rates aim to see the situation when \(x_k\) is close to \(x^*\) (so sometimes we call it “local convergence rate”)

Newton method: quadratic convergence