Spline Interpolation I

- Let

$$s_j(x_j) = f(x_j), j = 0, \ldots, n-1, \tag{1}$$
$$s_{n-1}(x_n) = f(x_n)$$
$$s_{j+1}(x_{j+1}) = s_j(x_{j+1}), j = 0, \ldots, n-2 \tag{2}$$
$$s'_{j+1}(x_{j+1}) = s'_j(x_{j+1}), j = 0, \ldots, n-2$$
$$s''_{j+1}(x_{j+1}) = s''_j(x_{j+1}), j = 0, \ldots, n-2$$

- Number of conditions

$$n + 1 + 3(n-1) = 4n - 2$$

Spline Interpolation II

- Boundary conditions

$$s_0''(x_0) = s_{n-1}''(x_n) = 0 \text{ or}$$
$$s_0'(x_0) = f'(x_0) \text{ and } s_{n-1}'(x_n) = f'(x_n)$$

- Total $4n$ conditions $\Rightarrow$ n cubic polynomials
- Construct cubic polynomials

$$s_j(x) \equiv a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3,$$
$$j = 0, \ldots, n - 1$$

Immediately

$$s_j(x_j) = a_j = f(x_j),\ j = 0, \ldots, n - 1$$

Spline Interpolation III

- Now (2) becomes

$$\begin{aligned} a_{j+1} &= s_j(x_{j+1}) \\ &= a_j + b_j(x_{j+1} - x_j) + c_j(x_{j+1} - x_j)^2 \\ &\quad +d_j(x_{j+1} - x_j)^3, j = 0, \ldots, n-2 \end{aligned}$$

- Define

$$\begin{aligned} h_j &\equiv x_{j+1} - x_j \\ a_n &\equiv f(x_n) \end{aligned}$$

Earlier a_j is only from 0 to $n-1$

Spline Interpolation IV

We have

$$\begin{aligned} & a_n = s_{n-1}(x_n) \\ = \; & a_{n-1} + b_{n-1}(x_n - x_{n-1}) + c_j(x_n - x_{n-1})^2 + \\ & d_j(x_n - x_{n-1})^3 \end{aligned}$$

Then

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3, \quad j = 0, \ldots, n-1 \tag{3}$$

Spline Interpolation V

- Define

$$b_n \equiv s'_{n-1}(x_n) \tag{4}$$

Earlier b_j is only 0 to $n-1$
Using

$$s'_j(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$$
$$s'_{j+1}(x_{j+1}) = s'_j(x_{j+1}), \quad j = 0, \ldots, n-2$$
$$b_{j+1} = b_j + 2c_jh_j + 3d_jh_j^2, \quad j = 0, \ldots, n-1 \tag{5}$$

Spline Interpolation VI

- Define

$$c_n \equiv \frac{s''_{n-1}(x_n)}{2}$$

Since

$$\begin{aligned} s''_{j+1}(x_{j+1}) &= s''_j(x_{j+1}), \quad j = 0, \ldots, n-2 \\ s''_j(x) &= 2c_j + 6d_j(x - x_j) \\ s''_j(x_{j+1}) &= 2c_j + 6d_j h_j \\ s''_{j+1}(x_{j+1}) &= 2c_{j+1} \end{aligned}$$

Spline Interpolation VII

we have

$$c_{j+1} = c_j + 3d_j(h_j), \quad j = 0, \ldots, n-1$$

- Thus

$$d_j = \frac{c_{j+1} - c_j}{3h_j}$$

Spline Interpolation VIII

so (3) becomes

$$
\begin{aligned}
& a_{j+1} \\
= \ & a_j + b_j h_j + c_j h_j^2 + \frac{c_{j+1} - c_j}{3h_j} h_j^3 \\
= \ & a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}), \quad j = 0, \ldots, n-1 \qquad (6)
\end{aligned}
$$

Spline Interpolation IX

- (5) becomes

$$
\begin{aligned}
b_{j+1} &= b_j + 2c_j h_j + 3d_j h_j^2 \\
&= b_j + 2c_j h_j + 3\frac{c_{j+1} - c_j}{3h_j} h_j^2 \\
&= b_j + 2c_j h_j + h_j(c_{j+1} - c_j) \\
&= b_j + h_j(c_j + c_{j+1}), \quad j = 0, \ldots, n-1 \quad (7)
\end{aligned}
$$

- We have known a_j, unknowns: b_j, c_j

Spline Interpolation X

- From (6)

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), j = 0, \ldots, n-1$$

That is

$$b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j), j = 1, \ldots, n$$

Rewriting (7) to

$$b_j = b_{j-1} + h_{j-1}(c_{j-1} + c_j), , j = 1, \ldots, n$$

Spline Interpolation XI

and substituting things to it

$$\begin{aligned} & \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) \\ = \ & \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j) + \\ & h_{j-1}(c_{j-1} + c_j), j = 1, \ldots, n-1 \end{aligned}$$

- Note that j cannot be zero now

Spline Interpolation XII

- Finally

$$
\begin{aligned}
& \frac{h_{j-1}}{3} c_{j-1} + \frac{2(h_{j-1} + h_j)}{3} c_j + \frac{h_j}{3} c_{j+1} \\
= \; & \frac{1}{h_j}(a_{j+1} - a_j) - \frac{1}{h_{j-1}}(a_j - a_{j-1})
\end{aligned}
$$

so

$$
\begin{aligned}
& h_{j-1} c_{j-1} + 2(h_{j-1} + h_j) c_j + h_j c_{j+1} \\
= \; & \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}), j = 1, \ldots, n - 1 \quad (8)
\end{aligned}
$$

Spline Interpolation XIII

- Only unknowns: $c_j, j = 0, \ldots, n$
- Once we know c_j, then we can find b_j and d_j
- Boundary conditions
- If $s''_0(x_0) = s''_{n-1}(x_n) = 0$ then

$$\begin{aligned} s''_0(x) &= 2c_0 + 6d_0(x - x_0) \\ s''_0(x_0) &= 2c_0 = 0, \end{aligned}$$

and by the definition of c_n

$$s''_{n-1}(x_n) = 2c_n = 0$$

Spline Interpolation XIV

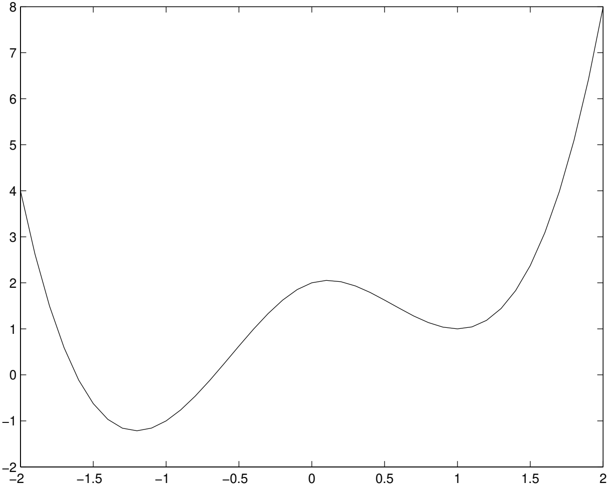
- Solve $c_j, j = 0, \ldots, n$ using $(n + 1)$ equalities: (8) and the above two equations
- We can use other boundary conditions such as

$$s_0'(x_0) = f'(x_0) \text{ and } s_{n-1}'(x_n) = f'(x_n)$$

- Example:

```
>> x = [-2 -1 0 1 2]';
>> y = [4 -1 2 1 8]';
>> yy=interp1(x,y,-2:0.1:2,'spline') ;
>> plot(-2:0.1:2,yy);
```

Spline Interpolation XV



Spline Interpolation XVI

```
>> [sp] = spline(x,y)

sp =

      form: 'pp'
    breaks: [-2 -1 0 1 2]
     coefs: [4x4 double]
    pieces: 4
     order: 4
       dim: 1
```

- `spline`: returns the spline structure

  `interp1`: 1-D interpolation and returns the approximate function value