Matrix-product Form of FFT I

- For the example,

$$A_3A_2A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \delta^2 & -\delta^2 & \delta^1 & -\delta^1 & \delta^3 & -\delta^3 \\ 1 & 1 & -1 & -1 & \delta^2 & \delta^2 & -\delta^2 & -\delta^2 \\ 1 & -1 & -\delta^2 & \delta^2 & \delta^3 & -\delta^3 & \delta^1 & -\delta^1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & \delta^2 & -\delta^2 & -\delta^1 & \delta^1 & -\delta^3 & +\delta^3 \\ 1 & 1 & -1 & -1 & -\delta^2 & -\delta^2 & \delta^2 & \delta^2 \\ 1 & -1 & -\delta^2 & \delta^2 & -\delta^3 & \delta^3 & -\delta^1 & \delta^1 \end{bmatrix}$$

Matrix-product Form of FFT II

- We can see that

$$A_3 A_2 A_1 P,$$

where

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & & & & & & & \\ & & & & 1 & & & \\ & & 1 & & & & & \\ & & & & & & 1 & \\ & 1 & & & & & & \\ & & & & & 1 & & \\ & & & 1 & & & & \\ & & & & & & & 1 \end{pmatrix} \end{array}$$

Matrix-product Form of FFT III

goes back to F obtained earlier:

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \delta^1 & \delta^2 & \delta^3 & -1 & -\delta^1 & -\delta^2 & -\delta^3 \\ 1 & \delta^2 & -1 & -\delta^2 & 1 & \delta^2 & -1 & -\delta^2 \\ 1 & \delta^3 & -\delta^2 & \delta & -1 & -\delta^3 & \delta^2 & -\delta^1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\delta^1 & \delta^2 & -\delta^3 & -1 & \delta^1 & -\delta^2 & \delta^3 \\ 1 & -\delta^2 & -1 & \delta^2 & 1 & -\delta^2 & -1 & \delta^2 \\ 1 & -\delta^3 & -\delta^2 & -\delta & -1 & \delta^3 & \delta^2 & \delta^1 \end{bmatrix}$$

Matrix-product Form of FFT IV

- We see that P is a permutation matrix by reversing the "binary digits" of indices

0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

- $A_t \cdots A_1 P$ permutes columns of $A_t \cdots A_1$

Matrix-product Form of FFT V

- We explain below why P is obtained from the reverse of the binary digits

$$F_8 y = \underbrace{\begin{bmatrix} I_4 & \Omega_4 \\ I_4 & -\Omega_4 \end{bmatrix}}_{A_3 = B_8} \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix} \begin{bmatrix} y(0:2:7) \\ y(1:2:7) \end{bmatrix}$$

$$= \begin{bmatrix} I_4 & \Omega_4 \\ I_4 & -\Omega_4 \end{bmatrix} \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} y$$

Matrix-product Form of FFT VI

- We see that y is rearranged by the following way:

$$\begin{array}{ll} 0 & 000 \\ 1 & 001 \\ 2 & 010 \\ 3 & 011 \\ 4 & 100 \\ 5 & 101 \\ 6 & 110 \\ 7 & 111 \end{array} \Rightarrow \begin{array}{ll} 0 & 000 \\ 2 & 010 \\ 4 & 100 \\ 6 & 110 \\ 1 & 001 \\ 3 & 011 \\ 5 & 101 \\ 7 & 111 \end{array}$$

Matrix-product Form of FFT VII

- This is like that the binary representation is rotated to the left by one digit
- For example,

$$5 \Rightarrow 2(5-4)+1=3$$

$5-4$:	First digit "1" removed
$2(5-4)$:	Second, third digits shifted left
$2(5-4)+1$:	Add the original first digit "1" to the now last digit

Matrix-product Form of FFT VIII

- Now we can see that besides the last digit, the first two digits of both the upper and the lower parts are the same:

$$\begin{array}{cc} 0 & 00 \\ 1 & 01 \\ 2 & 10 \\ 3 & 11 \end{array}$$

- It is like that now we have a new

$$F_4 \tilde{y}$$

- By the same setting we should have

Matrix-product Form of FFT IX

$$F_8 y = A_3 \begin{bmatrix} I_2 & \Omega_2 & 0 & 0 \\ I_2 & -\Omega_2 & 0 & 0 \\ 0 & 0 & I_2 & \Omega_2 \\ 0 & 0 & I_2 & -\Omega_2 \end{bmatrix} \begin{bmatrix} F_2 & 0 & 0 & 0 \\ 0 & F_2 & 0 & 0 \\ 0 & 0 & F_2 & 0 \\ 0 & 0 & 0 & F_2 \end{bmatrix}$$

$$\times$$

Matrix-product Form of FFT X

$$\underbrace{\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}}_{\text{block diagonal}} \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} y$$

Matrix-product Form of FFT XI

- We have

$$A_2 = \begin{bmatrix} I_2 & \Omega_2 & 0 & 0 \\ I_2 & -\Omega_2 & 0 & 0 \\ 0 & 0 & I_2 & \Omega_2 \\ 0 & 0 & I_2 & -\Omega_2 \end{bmatrix}$$

- For the next step, the permutation matrix is I. Thus

Matrix-product Form of FFT XII

$$P = I_8 \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

Matrix-product Form of FFT XIII

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Matrix-product Form of FFT XIV

and

$$A_1 = \begin{bmatrix} I_1 & \Omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_1 & -\Omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_1 & \Omega_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_1 & -\Omega_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_1 & \Omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_1 & -\Omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_1 & \Omega_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_1 & -\Omega_1 \end{bmatrix}$$

Therefore, y is arranged by the following way

Matrix-product Form of FFT XV

$$
\begin{array}{ll}
0 & 000 \\
1 & 001 \\
2 & 010 \\
3 & 011 \\
4 & 100 \\
5 & 101 \\
6 & 110 \\
7 & 111
\end{array}
\Rightarrow
\begin{array}{ll}
0 & 000 \\
2 & 010 \\
4 & 100 \\
6 & 110 \\
1 & 001 \\
3 & 011 \\
5 & 101 \\
7 & 111
\end{array}
\Rightarrow
\begin{array}{ll}
0 & 000 \\
4 & 100 \\
2 & 010 \\
6 & 110 \\
1 & 001 \\
5 & 101 \\
3 & 011 \\
7 & 111
\end{array}
$$

Thus we have explained the reason of using the reverse of the binary digits