Recursive Form of FFT I

- Next we follow the discussion in Section 4.6.4 of the book Matrix Computation
- Consider the case of $m = 4$. Then

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \delta^1 & \delta^2 & \delta^3 & \delta^4 & \delta^5 & \delta^6 & \delta^7 \\ 1 & \delta^2 & \delta^4 & \delta^6 & 1 & \delta^2 & \delta^4 & \delta^6 \\ 1 & \delta^3 & \delta^6 & \delta^1 & \delta^4 & \delta^7 & \delta^2 & \delta^5 \\ 1 & \delta^4 & 1 & \delta^4 & 1 & \delta^4 & 1 & \delta^4 \\ 1 & \delta^5 & \delta^2 & \delta^7 & \delta^4 & \delta^1 & \delta^6 & \delta^3 \\ 1 & \delta^6 & \delta^4 & \delta^2 & 1 & \delta^6 & \delta^4 & \delta^2 \\ 1 & \delta^7 & \delta^6 & \delta^5 & \delta^4 & \delta^3 & \delta^2 & \delta^1 \end{bmatrix}$$

Recursive Form of FFT II

where

$$\delta = e^{-i\pi/4}, \delta^8 = 1, \delta^4 = -1$$

Recursive Form of FFT III

This can be written as

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \delta^1 & \delta^2 & \delta^3 & -1 & -\delta^1 & -\delta^2 & -\delta^3 \\ 1 & \delta^2 & -1 & -\delta^2 & 1 & \delta^2 & -1 & -\delta^2 \\ 1 & \delta^3 & -\delta^2 & \delta & -1 & -\delta^3 & \delta^2 & -\delta^1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\delta^1 & \delta^2 & -\delta^3 & -1 & \delta^1 & -\delta^2 & \delta^3 \\ 1 & -\delta^2 & -1 & \delta^2 & 1 & -\delta^2 & -1 & \delta^2 \\ 1 & -\delta^3 & -\delta^2 & -\delta & -1 & \delta^3 & \delta^2 & \delta^1 \end{bmatrix} \tag{1}$$

Recursive Form of FFT IV

- Suppose

$$c = \begin{bmatrix} 0 & 2 & 4 & 6 & 1 & 3 & 5 & 7 \end{bmatrix}$$

  is an index vector
- Then

$$Fy = F(:, c)y(c)$$

  and

$$F_8(:, c) = \begin{bmatrix} F_4 & \Omega_4 F_4 \\ F_4 & -\Omega_4 F_4 \end{bmatrix}$$

Recursive Form of FFT V

- We call the current F as F_8. Then

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \delta^2 & -1 & -\delta^2 \\ 1 & -1 & 1 & -1 \\ 1 & -\delta^2 & -1 & \delta^2 \end{bmatrix}$$

  and

$$\Omega_4 = \begin{bmatrix} 1 & & & \\ & \delta & & \\ & & \delta^2 & \\ & & & \delta^3 \end{bmatrix}$$

Recursive Form of FFT VI

- We have

$$\Omega_4 F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \delta^1 & \delta^3 & -\delta^1 & -\delta^3 \\ \delta^2 & -\delta^2 & \delta^2 & -\delta^2 \\ \delta^3 & \delta^1 & -\delta^3 & -\delta^1 \end{bmatrix}$$

- Then

$$\begin{aligned} Fy &= F(:, c)y(c) \\ &= \begin{bmatrix} F_4 & \Omega_4 F_4 \\ F_4 & -\Omega_4 F_4 \end{bmatrix} \begin{bmatrix} y(0:2:7) \\ y(1:2:7) \end{bmatrix} \\ &= \begin{bmatrix} I & \Omega_4 \\ I & -\Omega_4 \end{bmatrix} \begin{bmatrix} F_4 y(0:2:7) \\ F_4 y(1:2:7) \end{bmatrix} \end{aligned}$$

Recursive Form of FFT VII

- We can use the same way to calculate

$$F_4 y(0:2:7) \text{ and } F_4 y(1:2:7) \tag{2}$$

- For $F_8 y$, multiplying with I or Ω takes $O(m)$ time
- Then for (2), the cost is also $O(m)$
- Thus the cost at each step is $O(m)$
- The number of steps is $O(\log m)$
- Total cost

$$O(m \log m)$$

Matrix-product Form of FFT I

- In practice, FFT is done by matrix products rather than a recursive implementation
- Assume

$$m = 2^?$$

- We have

$$F = A_t \cdots A_1 P$$

Matrix-product Form of FFT II

where

$$t = \log 2m$$
$$t - 1 = \log m$$
$$\vdots$$

and P is a permutation matrix

$$A_t = I_r \otimes B_L,$$

where

$$L = 2^t, r = 2m/L$$

I_r is an $r \times r$ identity matrix

Matrix-product Form of FFT III

- Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ & \vdots & & \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

- Details of B_L

$$B_L = \begin{bmatrix} I_{L/2} & \Omega_{L/2} \\ I_{L/2} & -\Omega_{L/2} \end{bmatrix},$$

$$\Omega_{L/2} = \text{diag}(1, \delta^r, \ldots, (\delta^r)^{L/2-1})$$

Matrix-product Form of FFT IV

- $t = 3$

$$L = 2, m = 8, r = 1, l_r = 1,$$

$$B_8 = \begin{bmatrix} 1 & & & & 1 & & & \\ & 1 & & & & \delta & & \\ & & 1 & & & & \delta^2 & \\ & & & 1 & & & & \delta^3 \\ 1 & & & & -1 & & & \\ & 1 & & & & -\delta & & \\ & & 1 & & & & -\delta^2 & \\ & & & 1 & & & & -\delta^3 \end{bmatrix}$$

Matrix-product Form of FFT V

- $t = 2$

$$L = 4, r = 2, I_r = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & \delta^2 \\ 1 & & -1 & \\ & 1 & & -\delta^2 \end{bmatrix}$$

- $t = 1$

$$L = 2, r = 4, B_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Matrix-product Form of FFT VI

The product

$$
\begin{aligned}
& A_3 A_2 A_1 \\
= & B_8 \begin{bmatrix} B_4 & \\ & B_4 \end{bmatrix} \begin{bmatrix} B_2 & & & \\ & B_2 & & \\ & & B_2 & \\ & & & B_2 \end{bmatrix}
\end{aligned}
$$

Matrix-product Form of FFT VII

$$B_4 \begin{bmatrix} B_2 & \\ & B_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & 1 & \\ & 1 & & \delta^2 \\ 1 & & -1 & \\ & 1 & & -\delta^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \delta^2 & -\delta^2 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\delta^2 & \delta^2 \end{bmatrix}$$

Matrix-product Form of FFT VIII

- Finally,

$$A_3A_2A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \delta^2 & -\delta^2 & \delta^1 & -\delta^1 & \delta^3 & -\delta^3 \\ 1 & 1 & -1 & -1 & \delta^2 & \delta^2 & -\delta^2 & -\delta^2 \\ 1 & -1 & -\delta^2 & \delta^2 & \delta^3 & -\delta^3 & \delta^1 & -\delta^1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & \delta^2 & -\delta^2 & -\delta^1 & \delta^1 & -\delta^3 & +\delta^3 \\ 1 & 1 & -1 & -1 & -\delta^2 & -\delta^2 & \delta^2 & \delta^2 \\ 1 & -1 & -\delta^2 & \delta^2 & -\delta^3 & \delta^3 & -\delta^1 & \delta^1 \end{bmatrix}$$

We can see that its columns are a permutation of those in F in (1)

Matrix-product Form of FFT IX

- Next we will discuss details of the permutation