Numerical Methods 2021 — Midterm 1

Solutions

Problem 1 (5 pts). Give the binary representation of -528.3125 under the IEEE standard.

Solution.

$$528.3125(\text{base } 10) = 1.0000100000101 \times 2^9(\text{base } 2)$$

$$1 \quad 10000001000 \quad 0000100000101\underbrace{0\cdots 0}_{\text{40 zeros}}$$

Problem 2 (35 pts). Consider a floating-point system with base β and precision p. In the class, we show that the relative rounding error is bounded after using 1 additional guard digit. The original theorem is stated below:

$$\text{Using } p+1 \text{ digits for } x-y \Rightarrow \text{relative rounding error} < 2\epsilon \ (\epsilon : \text{machine epsilon}) \tag{1}$$

In our proof, the procedure to calculate $x-y$ is showed below. For example, when $\beta = 10$ and $p = 3$:

- Step 1: Given $x = \underbrace{2.03}_{p \text{ digits}}$ and $y = \underbrace{1.51}_{p \text{ digits}} \times 10^{-3}$.
- Step 2: Shift y to the same base of x, $y = 0.00151$
- Step 3: **Truncate** y to $\bar{y}$ with $p+1$ precision, $\bar{y} = \underbrace{0.001}_{p+1 \text{ digits}}$.
- Step 4: $z = x - \bar{y} = \underbrace{2.029}_{p+1 \text{ digits}}$.
- Step 5: Round z to $\bar{z}$ with p precision, $\bar{z} = 2.03$. The $\bar{z}$ is the result of operation.

(a) (15 pts) In the proof of (1), we give a bound of $|y - \bar{y}|$

$$|y - \bar{y}| < (\beta - 1)(\beta^{-(p+1)} + \beta^{-(p+2)} + \cdots + \beta^{-(p+k)})$$

by assuming the truncation. But if we change truncation to **rounding odd**, then the bound in (1) can be tighter. Besides, in the Step 5 we also consider using **rounding odd**. (**Rounding odd** is round to nearest value if there is only one nearest value. When two nearest values exist, you must also consider previous digit. If previous digit is even, round up. Otherwise, round down. For example, when $\beta = 10$ and $p = 1$, $1.5 \underset{\text{round}}{\rightarrow} 1$, $1.51 \underset{\text{round}}{\rightarrow} 2$, $1.49 \underset{\text{round}}{\rightarrow} 1$ and $2.5 \underset{\text{round}}{\rightarrow} 3$.)

Redo the proof by showing that

$$\text{relative rounding error} \leq (1 + \frac{1}{\beta})\epsilon$$

You can prove the theorem by discussing the following cases.

- Case 1: Show that when $x - y \geq 1$,

$$\text{rel. rounding error} \leq (1 + \frac{1}{\beta})\epsilon.$$

- Case 2: Show that when $x - \bar{y} \leq 1$,

$$\text{rel. rounding error} \leq \epsilon.$$

- Case 3: Show that when $x - y < 1$ and $x - \bar{y} > 1$, this case is impossible to happen.

Note that all details must be given.

(b) (10 pts) Consider $p = 3$ and $\beta = 10$ and assume $x = x_0.x_1 \ldots x_{p-1} \times \beta^0$. Give examples respectively leading to **the largest** absolute rounding error of Case 1 and Case 2. Note that here we check **absolute rounding error** instead of relative error. You must explain why your given x and y lead to the largest error.

(c) (10 pts) If 2 guard digits are used, recalculate your examples in (b) and compare your error with the one using exact computation then round.

Solution.

(a) WLOG, let $x > y$,

$$x = x_0.x_1x_2x_3 \cdots x_{p-2}x_{p-1} \tag{2}$$

and

$$y = 0.0 \cdots 0y_k \cdots y_{p-1}y_p \cdots y_{k+(p-1)}.$$

First we need to round y to $\bar{y}$ with $p+1$ precision. After rounding odd is used, one can see that

$$\bar{y} = y + \delta_1, \text{ where } |\delta_1| \leq \frac{\beta}{2}\beta^{-(p+1)} = \frac{1}{\beta}\epsilon. \tag{3}$$

Calculate

$$z = x - \bar{y}$$

and round odd z to $\bar{z}$ with p precision. The equality between z and $\bar{z}$ is

$$\bar{z} = z + \delta_2, \text{ where } |\delta_2| \leq \frac{\beta}{2}\beta^{-p} = \epsilon. \tag{4}$$

The error between $x - y$ and $\bar{z}$ is

$$\begin{aligned} \text{error} = |x - y - \bar{z}| &= |x - y - (z + \delta_2)| \\ &= |x - y - (x - \bar{y} + \delta_2)| \\ &= |x - y - (x - (y + \delta_1) + \delta_2)| \\ &= |x - y - x + (y + \delta_1) - \delta_2| \\ &= |\delta_1 - \delta_2|. \end{aligned} \tag{5}$$

- Case 1 $x - y \geq 1$:

$$\text{rel. rounding error} = \frac{\text{error}}{x - y} \leq \frac{|\delta_1 - \delta_2|}{1} \leq |\delta_1| + |\delta_2| = (1 + \frac{1}{\beta})\epsilon$$

- Case 2 $x - \bar{y} \leq 1$: In this case, $\delta_2 = 0$ because z can store all its digits within p precision. The relative rounding error is
$$\text{rel. rounding error} = \frac{|\delta_1|}{x-y}.$$
The smallest $x - y$ happens when x is as small as possible and y is as large as possible. The smallest possible x is
$$x = 1.000\cdots000$$
and the largest possible y is
$$y = 0.0\cdots0y_k y_{k+1}\cdots y_{k+p-1},$$
where $y_i = \beta - 1 \,\forall i = \{k, \cdots, k+(p-1)\}$. Therefore, given any x and y, the difference between them is
$$x - y \geq (\beta-1)(\beta^{-1} + \beta^{-2} + \cdots + \beta^{-(k-1)}) \tag{6}$$
From (6),
$$\begin{aligned}\text{rel. rounding error} &= \frac{|\delta_1|}{x-y} \\ &\leq \frac{\frac{1}{\beta}\epsilon}{(\beta-1)(\beta^{-1} + \beta^{-2} + \cdots + \beta^{-(k-1)})} \\ &\leq \epsilon\frac{1}{(\beta-1)(1 + \beta^{-1} + \cdots + \beta^{-(k-2)})} \\ &< \epsilon\end{aligned}$$
The last inequality comes from
$$\frac{1}{(\beta-1)(1 + \beta^{-1} + \cdots + \beta^{-(k-2)})} < \frac{1}{\beta-1} \leq 1.$$
- Case 3 $x - \bar{y} > 1$ and $x - y < 1$: We show that this case is impossible. Given $x - \bar{y} > 1$, this implies that
$$x - \bar{y} \geq 1.\underbrace{0\cdots0}_{p-1 \text{ zeros}}1.$$
Because
$$|y - \bar{y}| \leq \frac{\beta}{2}\beta^{-(p+1)} = \frac{1}{2}\beta^{-p} < 0.\underbrace{0\cdots0}_{p-1 \text{ zeros}}1,$$
this implies that
$$x - y \geq 1.$$
Therefore, case 3 can not happen.

In the end, from case 1, 2, and 3. We conclude that that bound can be at least $(1 + \frac{1}{\beta})\epsilon$.

(b) **Solution is not unique.**

- Case 1: From (3), (4) and (5), we have
$$\text{error} = |\delta_1 - \delta_2| \leq |\delta_1| + |\delta_2| = 0.0055$$

The maximum absolute error in case 1 is 0.0055.

$$\begin{aligned}
&x = 1.01 \text{ and } y = 0.0055\\
&\bar{y} = 0.005\\
&z = x - \bar{y} = 1.005\\
&\bar{z} = 1.01\\
&\text{err} = |x - y - \bar{z}| = |1.0045 - 1.01| = 0.0055
\end{aligned}$$

- Case 2: Because $\delta_2 = 0$ in case 2,

$$\text{error} = |\delta_1 - \delta_2| = |\delta_1| = 0.0005$$

The maximum absolute error in case 2 is 0.0005.

$$\begin{aligned}
&x = 1.00 \text{ and } y = 0.0005\\
&\bar{y} = 0.001\\
&z = x - \bar{y} = 0.999\\
&\bar{z} = 9.99 \times 10^{-1}\\
&\text{err} = |x - y - \bar{z}| = |0.9995 - 0.999| = 0.0005
\end{aligned}$$

(c)
- Case 1: Use 2 gaurd digit.

$$\begin{aligned}
&x = 1.01 \text{ and } y = 0.0055\\
&\bar{y} = 0.0055\\
&z = x - \bar{y} = 1.0045\\
&\bar{z} = 1.00\\
&\text{err} = |x - y - \bar{z}| = |1.0045 - 1.01| = 0.0055
\end{aligned}$$

Use exact computation.

$$\begin{aligned}
&x = 1.01 \text{ and } y = 0.0055\\
&z = x - y = 1.0045\\
&\bar{z} = 1.00\\
&\text{err} = |x - y - \bar{z}| = |1.0045 - 1.01| = 0.0055
\end{aligned}$$

The error is the same between 2 guard digits and exact computation.

- Case 2: Use 2 guard digit.

$$\begin{aligned}
&x = 1.00 \text{ and } y = 0.0005\\
&\bar{y} = 0.0005\\
&z = x - \bar{y} = 0.9995\\
&\bar{z} = 9.99 \times 10^{-1}\\
&\text{err} = |x - y - \bar{z}| = |0.9995 - 0.999| = 0.0005
\end{aligned}$$

Use exact computation.

$$\begin{aligned}
&x = 1.00 \text{ and } y = 0.0005 \\
&z = x - y = 0.9995 \\
&\bar{z} = 9.99 \times 10^{-1} \\
&\text{err} = |x - y - \bar{z}| = |0.9995 - 0.999| = 0.0005
\end{aligned}$$

The error is the same between 2 guard digits and exact computation.

Problem 3 (25 pts). Consider the following matrix

$$\begin{bmatrix} -8 & 4 & -18 & 4 \\ 6 & -8 & 6 & -5 \\ -12 & 6 & 18 & -24 \\ 4 & -16 & -6 & 14 \end{bmatrix}$$

Conduct LU factorization with pivoting.

(a) (15 pts) Give

$$P_1, M_1, P_2, M_2, P_3, M_3$$

such that

$$M_3 P_3 M_2 P_2 M_1 P_1 A = U.$$

(b) (10 pts) What are P, L, U such that

$$PA = LU?$$

Hint: In your U, there is only one fraction number, while others are integers.

Solution.

(a) Step 1: The first pivot is -12, so we should switch the first and third rows.

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ P_1 A = \begin{bmatrix} -12 & 6 & 18 & -24 \\ 6 & -8 & 6 & -5 \\ -8 & 4 & -18 & 4 \\ 4 & -16 & -6 & 14 \end{bmatrix}$$

Step 2: Do the Gaussian elimination on the first column.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2/3 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1 \end{bmatrix}, \ M_1 P_1 A = \begin{bmatrix} -12 & 6 & 18 & -24 \\ 0 & -5 & 15 & -17 \\ 0 & 0 & -30 & 20 \\ 0 & -14 & 0 & 6 \end{bmatrix}$$

Step 3: The second pivot is -14, so we should switch the second and fourth rows.

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ P_2 M_1 P_1 A = \begin{bmatrix} -12 & 6 & 18 & -24 \\ 0 & -14 & 0 & 6 \\ 0 & 0 & -30 & 20 \\ 0 & -5 & 15 & -17 \end{bmatrix}$$

Step 4: Do the Gaussian elimination on the second column.

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -5/14 & 0 & 1 \end{bmatrix}, \; M_2P_2M_1P_1A = \begin{bmatrix} -12 & 6 & 18 & -24 \\ 0 & -14 & 0 & 6 \\ 0 & 0 & -30 & 20 \\ 0 & 0 & 15 & -134/7 \end{bmatrix}$$

Step 5: The third pivot is -30, so there is no rows need to switch.

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; P_3M_2P_2M_1P_1A = \begin{bmatrix} -12 & 6 & 18 & -24 \\ 0 & -14 & 0 & 6 \\ 0 & 0 & -30 & 20 \\ 0 & 0 & 15 & -134/7 \end{bmatrix}$$

Step 6: Do the Gaussian elimination on the third column.

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}, \; M_3P_3M_2P_2M_1P_1A = \begin{bmatrix} -12 & 6 & 18 & -24 \\ 0 & -14 & 0 & 6 \\ 0 & 0 & -30 & 20 \\ 0 & 0 & 0 & -64/7 \end{bmatrix}$$

(b) We have

$$P = P_3P_2P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$U = \begin{bmatrix} -12 & 6 & 18 & -24 \\ 0 & -14 & 0 & 6 \\ 0 & 0 & -30 & 20 \\ 0 & 0 & 0 & -64/7 \end{bmatrix}$$

by (a). We can calculate L by

$$\begin{bmatrix} (P_3P_2M_1^{-1})_{:,1} & (P_3M_2^{-1})_{:,2} & (M_3^{-1})_{:,3} & (M_4^{-1})_{:,4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 2/3 & 0 & 1 & 0 \\ -1/2 & 5/14 & -1/2 & 1 \end{bmatrix},$$

where

$$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 2/3 & 0 & 1 & 0 \\ -1/3 & 0 & 0 & 1 \end{bmatrix}, \; M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5/14 & 0 & 1 \end{bmatrix},$$

$$M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}, \; M_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 4 (15 pts). Assume we have a floating-point system with

$$\beta = 10 \text{ and } p = 3.$$

In the proof of a theorem we have checked what happened in calculating

$$x_0 = x, x_1 = (x_0 \oplus y) \ominus y, \ldots, x_n = (x_{n-1} \oplus y) \ominus y$$

by using the rounding up scheme.

(a) (5 pts) Consider the same

$$x = 1.00 \text{ and } y = 0.555$$

Analyze the results by assuming this **rounding down** strategy:

$$\begin{aligned} 0, 1, 2, 3, 4, 5 &\Rightarrow \text{down} \\ 6, 7, 8, 9 &\Rightarrow \text{up} \end{aligned}$$

(b) (10 pts) Under the same rounding strategy in (a), is the statement

$$x_1 = x_2 = \cdots, \ \forall x, y \tag{7}$$

correct? Prove it if you think (7) is true. Otherwise, please give an example show us (7) is false.

Solution.

(a)

$$\begin{aligned} x_0 =& 1.00 \\ x_1 =& (x_0 \oplus 0.555) \ominus 0.555 = 1.55 \ominus 0.555 = 0.995 \\ x_2 =& (x_1 \oplus 0.555) \ominus 0.555 = 1.55 \ominus 0.555 = 0.995 \end{aligned}$$

Therefore, $x_1 = x_2 = \cdots = x_n = \cdots = 0.995$

Common mistake: Many round 0.995 to 0.99. Note that it can be stored under the system.

(b) Let us take

$$x = 1.03, y = 0.555$$

Thus, the following sequence

$$\begin{aligned} x_1 =& (1.03 \oplus 0.555) \ominus 0.555 = 1.58 \ominus 0.555 = 1.02 \\ x_2 =& (1.02 \oplus 0.555) \ominus 0.555 = 1.57 \ominus 0.555 = 1.01 \\ x_3 =& (1.01 \oplus 0.555) \ominus 0.555 = 1.56 \ominus 0.555 = 1.00 \\ x_4 =& (1.00 \oplus 0.555) \ominus 0.555 = 1.55 \ominus 0.555 = 0.995 \\ x_5 =& (0.995 \oplus 0.555) \ominus 0.555 = 1.55 \ominus 0.555 = 0.995 \end{aligned}$$

implies that (7) is false.

Problem 5 (20 pts). Consider the following matrix:

$$C = \begin{bmatrix} 25 & -20 & -10 & 30 \\ -20 & 20 & 6 & -38 \\ -10 & 6 & 69 & -29 \\ 30 & -38 & -29 & 94 \end{bmatrix}$$

(a) (10 pts) Show by the definition

$$C \text{ is positive definite } \Leftrightarrow x^T C x > 0, \forall x \neq 0$$

that this matrix is only positive semi-definite but not positive definite.

(b) (10 pts) Do the outer-product form of Cholesky factorization and show when/where it fails.

Solution.

(a) From the result in (b), it is sufficient to find

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 5 & -4 & -2 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 8 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{L^T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is,

$$\left\{ \begin{aligned} 5x_1 - 4x_2 - 2x_3 + 6x_4 &= 0 \\ 2x_2 - 1x_3 - 7x_4 &= 0 \\ 8x_3 - 3x_4 &= 0 \end{aligned} \right. \tag{8}$$

Let us take

$$x_3 = \frac{3}{8} x_4$$

so (8) is equivalent to

$$\left\{ \begin{aligned} 5x_1 - 4x_2 - 2 \cdot 3/8 x_4 + 6x_4 &= 0 \\ 2x_2 - 1 \cdot 3/8 x_4 - 7x_4 &= 0 \\ x_3 &= 3/8 \cdot x_4 \end{aligned} \right. \equiv \left\{ \begin{aligned} 5x_1 - 4x_2 + 21/4 \cdot x_4 &= 0 \\ 2x_2 - 59/8 \cdot x_4 &= 0 \\ x_3 &= 3/8 \cdot x_4 \end{aligned} \right. \tag{9}$$

Thus, we take

$$x_2 = \frac{59}{16} x_4$$

and (9) is equivalent to

$$\left\{ \begin{aligned} 5x_1 - 4 \cdot 59/16 \cdot x_4 + 21/4 \cdot x_4 &= 0 \\ x_2 &= 59/16 \cdot x_4 \\ x_3 &= 3/8 \cdot x_4 \end{aligned} \right. \equiv \left\{ \begin{aligned} 5x_1 - 19/2 \cdot x_4 &= 0 \\ x_2 &= 59/16 \cdot x_4 \\ x_3 &= 3/8 \cdot x_4 \end{aligned} \right. \tag{10}$$

By (10), we know that

$$Cx = 0$$

as

$$x = \begin{bmatrix} 19/10 \cdot t \\ 59/16 \cdot t \\ 3/8 \cdot t \\ t \end{bmatrix}, \ \forall t \neq 0$$

As we take

$$t = 80,$$

we have

$$x = [152, 295, 30, 80]^T$$

such that

$$x^T C x = 0.$$

Therefore, C is a semi-definite positive matrix.

(b) Let us take

$$\tilde{C}_1 = C$$

Step 1: The first pivot is 25, so we can take

$$l_1 = \frac{1}{\sqrt{25}} \cdot [25, -20, -10, -30]^T = [5, -4, -2, 6]^T.$$

Thereby,

$$\tilde{C}_2 = \tilde{C}_1 - l_1 \cdot l_1^T = \begin{bmatrix} 25 & -20 & -10 & 30 \\ -20 & 20 & 6 & -38 \\ -10 & 6 & 69 & -29 \\ 30 & -38 & -29 & 94 \end{bmatrix} - \begin{bmatrix} 25 & -20 & -10 & 30 \\ -20 & 16 & 8 & -24 \\ -10 & 8 & 4 & -12 \\ 30 & -24 & -12 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -14 \\ 0 & -2 & 65 & -17 \\ 0 & -14 & -17 & 58 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & -14 \\ 0 & -2 & 65 & -17 \\ 0 & -14 & -17 & 58 \end{bmatrix} \begin{bmatrix} 5 & -4 & -2 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2: The second pivot is 4, so we can take

$$l_2 = \frac{1}{\sqrt{4}} \cdot [0, 4, -2, -14]^T = [0, 2, -1, -7]^T.$$

Thus,

$$\tilde{C}_3 = \tilde{C}_2 - l_2 \cdot l_2^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -14 \\ 0 & -2 & 65 & -17 \\ 0 & -14 & -17 & 58 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -14 \\ 0 & -2 & 1 & 7 \\ 0 & -14 & 7 & 49 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 64 & -24 \\ 0 & 0 & -24 & 9 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 6 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 64 & -24 \\ 0 & 0 & -24 & 9 \end{bmatrix} \begin{bmatrix} 5 & -4 & -2 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 3: The third pivot is 64, so we can take

$$l_3 = \frac{1}{\sqrt{64}} \cdot [0, 0, 64, -24]^T = [0, 0, 8, -3]^T.$$

Thus,

$$\tilde{C}_4 = \tilde{C}_3 - l_3 \cdot l_3^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 64 & -24 \\ 0 & 0 & -24 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 64 & -24 \\ 0 & 0 & -24 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ -2 & -1 & 8 & 0 \\ 6 & -7 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & -4 & -2 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 8 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 4: The fourth pivot is 0, and we get an error!