Problem 1 (20 pts). Consider the following four pairs of \((x, f(x))\):

\((0, 3), (1, 0), (2, 3), (3, 0)\).

(a) (5 pts) Find the Lagrange Polynomial. You must do the calculation to obtain a final form of

\[a_3x^3 + a_2x^2 + a_1x + a_0.\]

(b) (15 pts) Find the spline by the following boundary conditions

\[s''_0(x_0) = 0, s''_3(x_3) = 0.\]

You must show details of every step in calculating \(a_j, b_j, c_j\) and \(d_j\).

Solution.

(a) Lagrange polynomial:

\[
P(x) = \sum_{k=0}^{3} L_{n,k}(x)f(x_k)
\]

\[
= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)
\]

\[
= \frac{(x-1)(x-2)(x-3)}{2} 3 + \frac{(x-0)(x-1)(x-3)}{2} 3
\]

\[
= -4x^3 + 18x^2 - 20x + 6
\]

\[
= -2x^3 + 9x^2 - 10x + 3
\]

(b) We have to determine \(a_j, b_j, c_j\) and \(d_j\) of

\[s_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3,\]

where \(j = 0, 1, 2\).
• Define \( h_j \)
  \[
  h_0 = x_1 - x_0 = 1, \quad h_1 = x_2 - x_1 = 1, \quad h_2 = x_3 - x_2 = 1.
  \]

• Compute \( a_j \)
  \[
  a_0 = f(0) = 3, \quad a_1 = f(1) = 0, \quad a_2 = f(2) = 3, \quad a_3 \equiv f(3) = 0.
  \]

• Compute \( c_j \) From the boundary condition,
  \[
  s_0''(0) = 2c_0 = 0 \Rightarrow c_0 = 0
  \]
  and
  \[
  c_3 \equiv \frac{s_2''(3)}{2} = 0.
  \]
  Compute \( c_1 \) and \( c_2 \) by
  \[
  h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_jc_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})
  \]
  where \( j = 1, 2 \). We have
  \[
  h_0c_0 + 2(h_0 + h_1)c_1 + h_1c_2 = \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0)
  \]
  \[
  h_1c_1 + 2(h_1 + h_2)c_2 + h_2c_3 = \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1).
  \]
  That is,
  \[
  4c_1 + c_2 = 3(a_2 - a_1) - 3(a_1 - a_0) = 3 \times 3 - 3 \times (-3) = 18
  \]
  \[
  c_1 + 4c_2 = 3(a_3 - a_2) - 3(a_2 - a_1) = 3 \times (-3) - 3 \times 3 = -18.
  \]
  We have
  \[
  c_1 = 6, c_2 = -6.
  \]

• Compute \( b_j \)
  \[
  b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}),
  \]
  where \( j = 0, 1, 2 \). We have
  \[
  b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1) = -3 - \frac{1}{3}(6) = -5
  \]
  \[
  b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(2c_1 + c_2) = 3 - \frac{1}{3}(6) = 1
  \]
  \[
  b_2 = \frac{1}{h_2}(a_3 - a_2) - \frac{h_2}{3}(2c_2 + c_3) = -3 - \frac{1}{3}(-12) = 1.
  \]

• Compute \( d_j \)
  \[
  d_j = \frac{c_{j+1} - c_j}{3h_j},
  \]
  where \( j = 0, 1, 2 \). We have
  \[
  d_0 = \frac{c_1 - c_0}{3h_0} = \frac{6}{3} = 2
  \]
  \[
  d_1 = \frac{c_2 - c_1}{3h_1} = \frac{-12}{3} = -4
  \]
  \[
  d_2 = \frac{c_3 - c_2}{3h_2} = \frac{6}{3} = 2
  \]
Finally, we have
\[
\begin{align*}
s_0(x) &= a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 \\
&= 3 + -5(x - 0) + 0(x - 0)^2 + 2(x - 0)^3 \\
&= 2x^3 - 5x + 3 \\
s_1(x) &= a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \\
&= 0 + 1(x - 1) + 6(x - 1)^2 + -4(x - 1)^3 \\
&= (x - 1) + 6(x - 1)^2 - 4(x - 1)^3 \\
&= -4x^3 + 18x^2 - 23x + 9 \\
s_2(x) &= a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \\
&= 3 + 1(x - 2) + -6(x - 2)^2 + 2(x - 2)^3 \\
&= 3 + (x - 2) - 6(x - 2)^2 + 2(x - 2)^3 \\
&= 2x^3 - 18x^2 + 49x - 39
\end{align*}
\]

**Problem 2 (10 pts).** Consider the following linear regression

\[
\min_a F(a) = \sum_{i=1}^{4} (a_0 + a_1x_{i1} + a_2x_{i2} - y_i)^2
\]

Give an example with
\[
x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} \neq x_j = \begin{bmatrix} x_{j1} \\ x_{j2} \end{bmatrix}, \forall i, j
\]
such that the resulting matrix in the linear system is positive semi-definite but not positive definite. You must also show the linear system and explain that it is only positive semi-definite.

**Solution.** Let us calculate the gradient as

\[
\nabla_a F = \sum_{i=1}^{4} \begin{bmatrix} \frac{2(a_0 + a_1 + x_{i1} + a_2x_{i2} - y_i)}{2(a_0 + a_1 + x_{i1} + a_2x_{i2} - y_i)x_{i1}} \\ \frac{2(a_0 + a_1 + x_{i1} + a_2x_{i2} - y_i)x_{i1}}{2(a_0 + a_1 + x_{i1} + a_2x_{i2} - y_i)x_{i2}} \end{bmatrix}
\]

and the Hessian matrix can be calculated as

\[
\nabla^2_a F = \sum_{i=1}^{4} \begin{bmatrix} \frac{2}{2x_{i1}} & \frac{2x_{i1}}{2x_{i1}} & \frac{2x_{i2}}{2x_{i1}} \\ \frac{2x_{i2}}{2x_{i2}} & \frac{2x_{i2}}{2x_{i2}} & \frac{2x_{i2}}{2x_{i2}} \end{bmatrix}
\]

The determinant of \( \nabla^2_a F \) is

\[
\frac{1}{8}(4S_{x_1x_1}S_{x_2x_2} + 2S_{x_1}S_{x_2}S_{x_1x_2} - S_{x_2}S_{x_2}S_{x_1x_1} - S_{x_1}S_{x_1}S_{x_2x_2} - 4S_{x_1x_2}S_{x_1x_2}) \quad (1)
\]

where

\[
S_{x_1} = \sum_{i=1}^{4} x_{is}
\]

\[
S_{x_1x_2} = \sum_{i=1}^{4} x_{is}x_{it}
\]
for all \( s, t \in \{1, 2\} \). We can take

\[
\begin{align*}
  x_1 &= (1, 1) \\
  x_2 &= (1, -1) \\
  x_3 &= (1, 2) \\
  x_4 &= (1, -2)
\end{align*}
\]

such that

\[
S_{x_2} = S_{x_1 x_2} = 0,
\]

and (1) is equal to

\[
\frac{1}{8} (4S_{x_1 x_1} S_{x_2 x_2} - S_{x_1 x_1} S_{x_2 x_2}) = \frac{1}{8} (4 \cdot 4 \cdot 10 - 4 \cdot 4 \cdot 10) = 0.
\]

Furthermore, since

\[
\nabla_2 a F = 2 \begin{bmatrix}
  4 & S_{x_1 x_1} & S_{x_2 x_2} \\
  S_{x_1 x_1} & S_{x_1 x_2} & S_{x_2 x_2}
\end{bmatrix} = 2 \begin{bmatrix}
  4 & 4 & 0 \\
  4 & 4 & 0 \\
  0 & 0 & 10
\end{bmatrix},
\]

we have

\[
a^T \nabla_2^2 F a = 2 \left( 4(a_0 + a_1)^2 + 10a_2^2 \right) \geq 0.
\]

Therefore, we can say that \( \nabla_2^2 F \) is positive semi-definite but not positive definite.

**Problem 3 (30 pts).** We mention that the most commonly used setting of spline is by piece-wise degree-3 polynomials, but now we are interested in using degree-4 polynomials. Naturally, we additionally consider

\[
s'''_{j+1}(x_{j+1}) = s'''_j(x_{j+1}), \; j = 0, \ldots, n - 2.
\]

(a) (5 pts) Without considering boundary conditions, what are number of equations and number of variables? You need to list those equations with the functions \( s_j \), for \( j = 0, \ldots, n - 1 \).

(b) (10 pts) Consider the definition of \( s_j(x) \)

\[
s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 + e_j(x - x_j)^4.
\]

Please list the equations in (a) with the variables \( a_j, b_j, c_j, d_j \) and \( e_j \). You can define

\[
h_j = x_{j+1} - x_j
\]

to simplify these equations.

(c) (15 pts) Please simplify from your equations in (b) by

i) Represent and substitute \( e_j \) with \( d_j \) and \( h_j \), for \( j = 0, \ldots, n - 1 \). Please show the equations after this simplification.

ii) Represent and substitute \( b_j \) with \( a_j, c_j, d_j \) and \( h_j \), for \( j = 0, \ldots, n - 1 \). Please show the equations after this simplification.

**Solution.**
(a) We have the following equations

\[ s_j(x_j) = f(x_j), \quad j = 0, \ldots, n - 1 \]  \hspace{1cm} (2)
\[ s_{n-1}(x_n) = f(x_n), \]  \hspace{1cm} (3)
\[ s_j(x_{j+1}) = s_{j+1}(x_{j+1}), \quad j = 0, \ldots, n - 2 \]  \hspace{1cm} (4)
\[ s'_j(x_{j+1}) = s'_{j+1}(x_{j+1}), \quad j = 0, \ldots, n - 2 \]  \hspace{1cm} (5)
\[ s''_j(x_{j+1}) = s''_{j+1}(x_{j+1}), \quad j = 0, \ldots, n - 2 \]  \hspace{1cm} (6)
\[ s'''_j(x_{j+1}) = s'''_{j+1}(x_{j+1}), \quad j = 0, \ldots, n - 2, \]  \hspace{1cm} (7)

so we have \( 5n - 3 \) equations. For the variables, our \( s_j(x) \) can be defined by

\[ a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 + e_j(x - x_j)^4. \]

Thus, we have \( 5n \) variables.

(b) Case 1: The equations (2) becomes

\[ a_j = f(x_j), \quad j = 0, \ldots, n - 1. \]

Case 2: The equation (3) becomes

\[ a_{n-1} + b_{n-1}h_{n-1} + c_{n-1}h_{n-1}^2 + d_{n-1}h_{n-1}^3 + e_{n-1}h_{n-1}^4 = f(x_n) \]

Case 3: The equations (4) becomes

\[ a_{j+1} = a_j + b_jh + c_jh^2 + d_jh^3 + e_jh^4, \quad j = 0, \ldots, n - 2. \]

Case 4: The equations (5) becomes

\[ b_{j+1} = b_j + 2c_jh + 3d_jh^2 + 4e_jh^3, \quad j = 0, \ldots, n - 2. \]

Case 5: The equations (6) becomes

\[ 2c_{j+1} = 2c_j + 6d_jh + 12e_jh^2, \quad j = 0, \ldots, n - 2. \]

Case 6: The equations (7) becomes

\[ 6d_{j+1} = 6d_j + 24e_jh, \quad j = 0, \ldots, n - 2. \]

(c) From (b)’s Case 6, we can find that

\[ e_j = \frac{d_{j+1} - d_j}{4h_j}, \quad j = 0, \ldots, n - 2. \]

Therefore, we can substitute the variable \( e_j \) in the other equations with \( d_j \) and \( d_{j+1} \). That is, we have

\[ a_{j+1} = a_j + b_jh + c_jh^2 + d_jh^3 + \frac{d_{j+1} - d_j}{4h_j}h^3, \quad j = 0, \ldots, n - 2 \]  \hspace{1cm} (8)
\[ b_{j+1} = b_j + 2c_jh + 3d_jh^2 + (d_{j+1} - d_j)h^3, \quad j = 0, \ldots, n - 2 \]
\[ 2c_{j+1} = 2c_j + 6d_jh + 3(d_{j+1} - d_j)h, \quad j = 0, \ldots, n - 2 \]
Next, we make (8) be

\[ b_j = \frac{a_{j+1} - a_j}{h_j} - c_j h_j - \frac{d_{j+1} + 3d_j}{4} h_j^2, \quad j = 0, \ldots, n - 2 \]

Thereby, we can substitute \( b_j \) in the equations with \( c_j \) and \( d_j \). That is, we have

\[ \frac{a_{j+2} - a_{j+1}}{h_{j+1}} - c_{j+1} h_{j+1} - \frac{d_{j+2} + 3d_{j+1}}{4} h_{j+1}^2 \]

\[ = \frac{a_{j+1} - a_j}{h_j} - c_j h_j - \frac{d_{j+1} + 3d_j}{4} h_j^2 + 2c_j h_j + 3d_j h_j^2 + (d_{j+1} - d_j) h_j^2, \quad j = 0, \ldots, n - 2 \]

\[ 2c_{j+1} - 2c_j + 6d_j h_j + 3(d_{j+1} - d_j) h_j, \quad j = 0, \ldots, n - 2 \]

**Problem 4 (10 pts).** Consider continuous least square. The function

\[ f(x) = x^2 \]

is approximated by

\[ P_1(x) = a_1 x + a_0 \]

over

\[ x \in [0, 1]. \]

Solve the linear system to get \( P_1 \).

**Solution.**

Our minimization problem is

\[
\min_a \int_0^1 [P_1(x) - f(x)]^2 dx
\]

\[
= \min_a \int_0^1 [a_1 x + a_0 - x^2]^2 dx
\]

\[
= \min_a \int_0^1 (x^4 + a_0^2 + a_1^2 x^2 - 2a_1 x^3 + 2a_1 a_0 x - 2a_0 x^2) dx
\]

\[
= \min_a \left( \int_0^1 x^4 dx + \int_0^1 a_0^2 dx + \int_0^1 a_1^2 x^2 dx - \int_0^1 2a_1 x^3 dx + \int_0^1 2a_1 a_0 x dx - \int_0^1 2a_0 x^2 dx \right)
\]

\[
= \min_a \left( \frac{1}{5} + a_0^2 + \frac{1}{3} a_1^2 - \frac{1}{2} a_1 + a_1 a_0 - \frac{2}{3} a_0 \right)
\]

Let us take

\[ g(a) = \left( \frac{1}{5} + a_0^2 + \frac{1}{3} a_1^2 - \frac{1}{2} a_1 + a_1 a_0 - \frac{2}{3} a_0 \right) \]

and utilize

\[ \nabla_a g = 0 \quad (9) \]

to get the solution. The linear system (9) is

\[
\begin{bmatrix}
2a_0 + a_1 - 2/3 \\
2/3 \cdot a_1 - 1/2 + a_0
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

After we solving it, we can get the solution

\[ a = \left( -\frac{1}{6}, 1 \right). \]
**Problem 5 (30 pts).** Given a function

\[ f(x) = \frac{2\pi}{x}, \]

approximate \( f(x) \) with a Fourier series with \( n \) term

\[ s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) \]

and \( 2m \) points

\[ (x_0, f(x_0)), \ldots, (x_{2m-1}, f(x_{2m-1})) \]

where

\[ x_k = -\pi + \frac{k}{m}. \]

(a) (15 pt) Given \( m = n = 2 \), in fast Fourier transform (FFT), we show that we can calculate \( c = F y \).

Show \( F \) and corresponding \( c \). (Hint: Euler’s formula: \( e^{ix} = \cos x + i \sin x \))

(b) (5 pt) Calculate

\[ a_0, a_1, a_2 \]

and

\[ b_1 \]

from \( c \).

(c) (10 pt) Decompose \( F \) to a sequence of matrix products,

\[ F = A_t \cdots A_1 P \]

**Solution.**

(a) Let

\[ \delta = e^{-i\pi} = e^{\frac{-i\pi}{2}} = \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) = -i. \]

The \( F \) is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \delta & \delta^2 & \delta^3 \\
1 & \delta^2 & \delta^4 & \delta^6 \\
1 & \delta^3 & \delta^6 & \delta^9
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \delta & -1 & -\delta \\
1 & -1 & 1 & -1 \\
1 & -\delta & -1 & \delta
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{bmatrix},
\]

because \( \delta = -i \).

The \( 2m \) points are

\[ (-\pi, -2), (-\frac{1}{2}\pi, -1), (0, 0), (\frac{1}{2}\pi, 1). \]

Then, we have

\[ y = \begin{bmatrix}
-2 \\
-1 \\
0 \\
1
\end{bmatrix} \]
and
\[
c = F y = \begin{bmatrix}
-2 \\
-2 + 2i \\
-2 \\
-2 - 2i
\end{bmatrix} = \begin{bmatrix}
-2 \\
-2 \\
-2 \\
-2
\end{bmatrix} + \begin{bmatrix}
0 \\
2 \\
0 \\
-2
\end{bmatrix} i
\]

(b) From
\[
a_k = \text{Re}\left(\frac{c_k(-1)^k}{m}\right) = \text{Re}\left(\frac{c_k(-1)^k}{2}\right)
\]
and
\[
b_k = -\text{Im}\left(\frac{c_k(-1)^k}{m}\right) = -\text{Im}\left(\frac{c_k(-1)^k}{2}\right).
\]
We have
\[
a_0 = \frac{-2(-1)^0}{2} = -1,
\]
\[
a_1 = \frac{-2(-1)^1}{2} = 1,
\]
\[
a_2 = \frac{-2(-1)^2}{2} = -1,
\]
and
\[
b_1 = \frac{-2(-1)^1}{2} = -(-1) = 1.
\]

(c) From
\[
t = \log 2m = \log 4 = 2,
\]
we know that
\[
F = A_2 A_1 P.
\]
Now, we derive $A_k$. When $k = 2$, we have
\[
L = 2^2 = 4, \quad r = \frac{2m}{L} = \frac{4}{4} = 1.
\]
and
\[
A_2 = I_r \otimes B_L = I_1 \otimes B_4 = 1 \otimes \begin{bmatrix}
I_2 & \Omega_2 \\
I_2 & -\Omega_2
\end{bmatrix}, \quad \text{where} \quad \Omega_2 = \begin{bmatrix}
1 & 0 \\
0 & \delta
\end{bmatrix}
\]
\[
= \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & \delta \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -\delta
\end{bmatrix}
\]
When $k = 1$, we have
\[
L = 2^1 = 2, \quad r = \frac{2m}{L} = 2.
\]
and

\[ A_1 = I_r \otimes B_L \]
\[ = I_2 \otimes B_2 \]
\[ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} I_1 & \Omega_1 \\ I_1 & -\Omega_1 \end{bmatrix}, \text{ where } \Omega_1 = [1] \]
\[ = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \]

The permutation is the reverse of each column binary representation.

- 00 → 00  column 0 swap to column 0
- 01 → 10  column 1 swap to column 2
- 10 → 01  column 2 swap to column 1
- 11 → 11  column 3 swap to column 3

Therefore, we have

\[ P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ F = A_2 A_1 P \]
\[ = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \delta \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\delta \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} P \]
\[ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \delta & -\delta \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\delta & \delta \end{bmatrix} P \]
\[ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \delta & -\delta \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\delta & \delta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]