• Please give details of your calculation. A direct answer without explanation is not counted.

• Your answers must be in English.

• Please carefully read problem statements.

• During the exam you are not allowed to borrow others’ class notes.

• Try to work on easier questions first.

1. (20%) If we define a matrix norm by

\[ \|A\| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2} \]

does it satisfy the three conditions needed by a norm:

\[ \|A\| \geq 0 \]
\[ \|A + B\| \leq \|A\| + \|B\| \]
\[ \|\alpha A\| = |\alpha|\|A\|, \]

where \( \alpha \) is a scalar. You need to check all three conditions.

2. (20%) Consider a matrix

\[ A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

(a) What is the 2-norm of this matrix? We mean the matrix norm defined in our lecture.

(b) What is the condition number of this matrix?

3. (10%)

(a) Give the compressed column format of the following matrix

\[ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(b) Give the compressed row format of the following matrix

\[ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]
Assume matlab (Fortran) arrays are used so array index starts from 1.

4. (30%) Assume $L \in \mathbb{R}^{n \times n}$ is a lower-triangular sparse matrix with $L_{ii} \neq 0$, and $x, b \in \mathbb{R}^{n \times 1}$ are stored as regular arrays. We would like to solve the linear system

$$Lx = b.$$ 

(a) Write the code if $L$ is stored in the compressed column format

(b) Write the code if $L$ is stored in the compressed row format

We assume that in the column (row) format, row (column) indices are sorted in ascending order.

5. (20%) Let $A$ be a $2 \times 2$ matrix with nonzero diagonal entries and given a linear system $Ax = b$. Does the convergence of Jacobi method implies $\rho(M^{-1}N) < 1$? Note that $M$ is a diagonal matrix containing $A$’s diagonal elements and $A = M - N$. 