E_{LBA} Undecidable I

- We have seen that

  A_{TM} undecidable
  A_{LBA} decidable

- However,

$$E_{\text{LBA}} = \{\langle M \rangle \mid M \text{ is an LBA where } L(M) = \emptyset\}$$

  is undecidable

- We do the proof by the computation history method
- Idea: the question of M accepts w can be solved by checking if $L(B) = \emptyset$, where B is an LBA

E_{LBA} Undecidable II

- Then because we assume E_{LBA} is decidable, we have a decider for A_{TM}
- The design of B: B recognizes all accepting computation histories for M on w

$$M \text{ accepts } w \Rightarrow L(B) \neq \emptyset$$
$$M \text{ rejects } w \Rightarrow L(B) = \emptyset$$

- We see that the machine is designed according to the given w. This strategy has been used in earlier examples
- Details of B: on any input x, we check if x is an accepting computation history for M on w

E_{LBA} Undecidable III

- Specifically, we check if x is

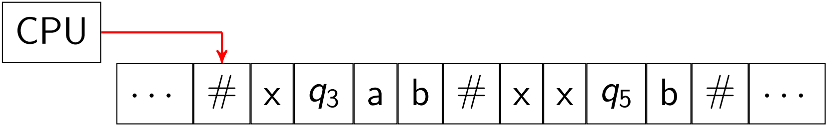
$$\#\underbrace{}_{C_1}\#\underbrace{}_{C_2}\#\cdots\#\underbrace{}_{C_l}\# \tag{1}$$

and $C_1, \ldots, C_l$ satisfy that

C_1 is the start configuration,
C_l is an accepting configuration, and
C_i follows from C_{i-1}

- The machine looks like

E_{LBA} Undecidable IV



- To begin, we check if the input x is in the form of (1)
- Next, C_1 is $q_0 w$, so checking the first condition is easy
- For the third condition, we scan if C_l contains q_{accept}
- Now C_i and C_{i+1} are the same except around the head position

E_{LBA} Undecidable V

- To compare C_i and C_{i+1}, the TM zigzags between them
- The setting looks good, but remember that B is an LBA
- The above discussion seems to show that our operations never go beyond $|x|$
- On the other hand, if you think extra space is needed, it is fine as long as the space needed is bounded by a constant factor of the length of $\#C_1\#\cdots\#C_l\#$

E_{LBA} Undecidable VI

- For example, if we copy C_i and C_{i+1} to the end for the comparison, the extra space needed is no more than $|\#C_1\# \cdots \#C_l\#|$
- Thus for input x we can check if the first half is $\#C_1\# \cdots \#C_l\#$
- Then the machine never goes beyond $|x|$. Further, if M accepts w, then at least one x is accepted by B

ALL_{CFG} Undecidable I

- Earlier we proved that

$$E_{\mathrm{CFG}} = \{\langle G \rangle \mid G : CFG, L(G) = \emptyset\}$$

  is decidable
- Now we show a related problem is undecidable

$$ALL_{\mathrm{CFG}} = \{\langle G \rangle \mid G : CFG, L(G) = \Sigma^*\}$$

- It checks if G generates all possible strings
- The proof is still by contradiction
  We assume ALL_{CFG} is decidable

ALL_{CFG} Undecidable II

- Idea: consider a CFG G such that

$$G \text{ generates } \Sigma^* \Leftrightarrow M \text{ does not accept } w$$

- This is equivalent to

$$\begin{cases} G \text{ generates } \Sigma^* & \text{if } M \text{ does not accept } w \\ G \text{ fails on some strings} & \text{if } M \text{ accepts } w \end{cases}$$

- If we have a decider on G, then we have a decider on A_{TM}
- If M accepts w, we let G fail to generate

ALL_{CFG} Undecidable III

an accepting computation history for M on w

- That is, for G, the input cannot be

$$\# \underbrace{\quad\quad}_{C_1} \# \underbrace{\quad\quad}_{C_2} \# \cdots \# \underbrace{\quad\quad}_{C_l} \#$$

where $C_1, \ldots, C_l$ satisfy that

C_1 is the start configuration,
C_l is an accepting configuration, and
C_i follows from C_{i-1}

- Therefore, G generates all strings
  1. that do not start with C_1,

ALL_{CFG} Undecidable IV

2. that do not end with an accepting configuration, or
3. C_i does not yield C_{i+1}

- Note that it's "or" because the opposite of

$$A \text{ and } B \text{ and } C$$

  is

$$\neg A \text{ or } \neg B \text{ or } \neg C$$

- On the other hand, if M does not accept w, no accepting computation history exists

ALL_{CFG} Undecidable V

Then G generates all strings

- But how to construct such a CFG?
- Let's generate an equivalent PDA
- The PDA nondeterministically checks three branches for the three requirements
- For example, the first branch checks if the beginning of the input is C_1 and accepts if it is not
- The third branch is more complicated
- It accepts if C_i does not properly yields C_{i+1}
- We can push C_i to stack ($\#$ allows us to extract C_i)

ALL_{CFG} Undecidable VI

- We pop the stack to compare C_i and C_{i+1}
- They are the same except around the head position
- A problem is that when we pop C_i, it is in the reverse order
- To enable the comparison, we write the accepting computation history differently

$$\# \underbrace{\quad \rightarrow \quad}_{C_1} \# \underbrace{\quad \leftarrow \quad}_{C_2^R} \# \underbrace{\quad \rightarrow \quad}_{C_3} \# \underbrace{\quad \leftarrow \quad}_{C_4^R} \# \cdots \# \underbrace{\qquad}_{C_l} \#$$

- By this way, when we pop C_2^R, we get C_2 and can do the comparison

ALL_{CFG} Undecidable VII

- This means that for any input x, if it is in the form of $\#C_1\# \cdots \#C_l\#$, we "treat" the second segment as C_2^R in designing operations