There quite a few undecidable problems

For example, program verification is in general not solvable

We will discuss an undecidable example called the “halting problem”
$A_{TM} = \{(M, w) \mid M: \text{a TM that accepts } w\}$

- We will prove that $A_{TM}$ is undecidable
- However, $A_{TM}$ is Turing recognizable
- We can simply simulate $\langle M, w \rangle$
- To be decidable we hope to avoid an infinite loop
  if at one point, know it cannot halt
  $\Rightarrow$ reject
- Thus this problem is called the halting problem
Diagonalization method

- We need a technique called “diagonalization method” for the proof.
- It was developed by Cantor in 1873 to check if two infinite sets are equal.
- Example: consider the set of even integers versus the set of \{0, 1\}^*.
- Both are infinite sets. Which one is larger?
- Definition: two sets are equal if elements can be paired.
Definition 4.12

- $f$ is a one-to-one function if:

\[ f(a) \neq f(b) \text{ if } a \neq b \]

- Left: a one-to-one function; right: not
Definition 4.12 II

- $f: A \rightarrow B$ onto if
  \[ \forall b \in B, \exists a \text{ such that } f(a) = b \]

- Example:
  \[ f(a) = a^2, \text{ where } A = (-\infty, \infty) \text{ and } B = (-\infty, \infty) \]

  This is not an onto function because for $b = -1$, there is no $a$ such that $f(a) = b$
However, if we change it to

\[ f(a) = a^2, \text{ where } A = (-\infty, \infty) \text{ and } B = [0, \infty) \]

it becomes an onto function

- **Definition:** a function is called a correspondence if it is one-to-one and onto

- **Example:**

\[ f(a) = a^3, \text{ where } A = (-\infty, \infty) \text{ and } B = (-\infty, \infty) \]
Thus a correspondence is a way of pairing elements of a set with elements of another.
Example 4.13

- \( N = \{1, 2, \ldots\} \)
- \( E = \{2, 4, \ldots\} \)
- The two sets can be paired

\[
\begin{array}{c|c}
 n & f(n) = 2n \\
\hline
1 & 2 \\
2 & 4 \\
\vdots & \vdots \\
\end{array}
\]

- We consider \( N \) and \( E \) have the same size
- Definition: a set is countable if it is finite or same size as \( N \)
Rational Numbers Countable I

- \( Q = \{ m/n \mid m, n \in \mathbb{N} \} \) countable

\[
\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
(Latex source from https://divisbyzero.com/2013/04/16/countability-of-the-rationals-drawn-using-tikz/)

- Note that we skip counting elements with common factors (e.g., 2/2)
Real Numbers not Countable

- We will use the diagonalization method.
- The proof is by contradiction.
- Assume $R$ is countable. Then there is a table as follows:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.14159 ...</td>
</tr>
<tr>
<td>2</td>
<td>55.55555 ...</td>
</tr>
<tr>
<td>3</td>
<td>0.12345 ...</td>
</tr>
<tr>
<td>4</td>
<td>0.50000 ...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Consider

\[ x = 0.4641 \ldots \]

\[ 4 \neq 1, 6 \neq 5 \]

We have

\[ x \neq f(n), \forall n \]

But \( x \in R \), so a contradiction

To avoid the problem

\[ 1 = 0.9999 \ldots \]

for every digit of \( x \) we should not choose 0 or 9