Mathematical notions I

- Set
  - Omitted
- Sequence and tuples
  - Sequence: Objects in order
    
    \[(7, 21, 57) \neq (57, 7, 21)\]
- Repetition

  set: \(\{7, 21, 57\} = \{7, 7, 21, 57\}\)

  sequences: \((7, 21, 57) \neq (7, 7, 21, 57)\)
Mathematical notions II

- **Tuples**: finite sequence
  
  \((7, 21, 57)\): 3-tuple

- **Cartesian product**:

  \[
  A = \{1, 2\}, \quad B = \{x, y\}
  \]

  \[
  A \times B = \{(1, x), (1, y), (2, x), (2, y)\}
  \]

- **Function**: single output

- **Relation**: scissors-paper-stone

  \[
  \begin{array}{c|ccc}
  \text{beats} & \text{scissors} & \text{paper} & \text{stone} \\
  \hline
  \text{scissors} & F & T & F \\
  \text{paper} & F & F & T \\
  \text{stone} & T & F & F \\
  \end{array}
  \]
Equivalence relation

1. reflexive
   \[ \forall x, xRx \]

2. symmetric
   \[ xRy \iff yRx \]

3. transitive
   \[ xRy, yRz \implies xRz \]

E.g. “=”
Example: $i \equiv_7 j$ if $0 = i - j \mod 7$

\[
i - i \mod 7 = 0
\]

\[
i - j = 7a, j - i = -7a
\]

\[
i - j = 7a, j - k = 7b
\]

\[
\Rightarrow i - k = 7(a + b)
\]

Graph

Undirected

Directed
Nodes (vertices)

- Edges: connection between nodes
- Degree = \# edges at a node

Subgraph: G is subgraph of H if
- G is a graph
- node(G) ⊂ node(H)
- edge(G) = subset of edge(H) connecting node(G)

In our example,
is a subgraph, but

is not

- **Strings and languages**
  - alphabet: \{0, 1\}
  - string: 1001
  - language: set of strings

- **Boolean logic**
  - true and false
0 (false) and 1 (true)
\[0 \land 0 = 0, \ 0 \lor 0 = 0, \ \neg 0 = 1\] (negation operation)
xor \[\otimes\]

\[0 \otimes 0 = 0\]
\[0 \otimes 1 = 1\]
\[1 \otimes 0 = 1\]
\[1 \otimes 1 = 0\]

implication
The above is called a truth table

Why

$$P = 0, \ Q = 1, \ \text{then} \ P \rightarrow Q = 1?$$

Consider

rainy $\rightarrow$ wet land
If not rainy, saying rainy implies wet land is ok.

\[ P \rightarrow Q \equiv \neg P \lor Q \]

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Proof I

- Direct proof:
  \[ A \rightarrow B \]

- Proof by contradiction
  \[ \neg B \rightarrow \neg A \]

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Example 1:
Every graph $\Rightarrow$ sum of degrees is even
- An example:
  \[ \circ - \circ - \circ \]
  \[ \# \text{ degrees} = 1 + 2 + 1 = 4 \]
- Each edge: 2 nodes
  \[ \text{total } \# \text{ degrees} = 2 \times \# \text{ edges} \]

What is the left side of the implication? It’s the definition of graphs

Example 2: $\sqrt{2}$ is irrational
Proof III

- The implication

  Definition of rational numbers

  \[ \Rightarrow \sqrt{2} \text{ is not rational} \]

  That is,

  If a rational number is ...

  \[ \Rightarrow \sqrt{2} \text{ is not rational} \]

  The opposite is

  If \( \sqrt{2} \) is rational

  \[ \Rightarrow \text{The rational number cannot be defined as } \ldots \]
By definition, $\sqrt{2}$ is rational means that

$$\sqrt{2} = \frac{m}{n}$$

and $m, n$ have no common factor

Then

$$2n^2 = m^2$$

Looks impossible. But how to write this formally?

First we prove that $m$ must be even. This is also proof by contradiction
If $m$ is not even, 

\[ m = 2k + 1. \]

Then 

\[ m^2 = 4(k^2 + k) + 1 \]

is not even and 

\[ m^2 = 2n^2 \]

does not hold.
Proof VI

Now suppose $m$ is even

$$m = 2k$$

Then

$$n^2 = 2k^2$$

By the same argument, $n$ is even

Thus $m, n$ have a common factor 2 and there is a contradiction