Polynomial vs. Exponential I

- Big difference
- $n^3 : n = 1000 \Rightarrow 10^9$
- $2^n : n = 1000 \Rightarrow 2^{1000} = 10^{1000 \log_{10} 2} \approx 10^{300} \gg 10^9$
- An algorithm with such complexity is not practical

Definition 7.2 I

- P: decidable languages in polynomial time on a deterministic (single-tape) TM

$$P = \cup_k \mathrm{TIME}(n^k).$$

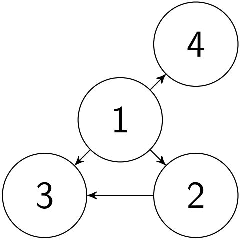
- How important this is ?

  P: "roughly" corresponds to problems solvable on a computer

PATH problem I

$$\text{PATH} = \{\langle G, s, t\rangle \mid G \text{ is a directed graph such that } \exists \text{ path from } s \text{ to } t\}$$

- Example:



PATH problem II

There is a path from $s = 1$ to $t = 3$

- We will prove that PATH $\in P$
- Let's start with a brute force way
  1. m: $\#$ nodes
  2. $|\text{path}| \leq m$
  3. $\#\text{paths} \leq m^m$
  4. sequentially check if one has s to t
- the cost is exponential
- A polynomial algorithm

  input $\langle G, s, t \rangle$, G includes nodes and edges

PATH problem III

1. mark s
2. repeat until no new node can be marked

   scan all edges, if for an edge $\langle a, b \rangle$:

   a is marked but b is not $\Rightarrow$ mark b
3. t marked $\Rightarrow$ accept

   otherwise $\Rightarrow$ reject

- \# of steps in the main loop: at most m (if no newly marked, stop)
- at each step, need to scan #edges$\leq m^2$
- cost to mark a node: polynomial
- whole algorithm: polynomial

Relatively Prime I

- x, y are relatively prime if they have no common (> 1) factors
- Example: 10 and 21

$$10 = 2 \times 5, 21 = 3 \times 7$$

- Example: 10 and 22

$$10 = 2 \times 5, 22 = 2 \times 11$$

  They are not relatively prime
- Problem: test if two numbers are relatively prime

Euclidean Algorithm I

- It can be used to find gcd (greatest common divisor)
- Example: gcd(18,24)=6
- We have

$$\text{gcd}(x, y) = 1 \Leftrightarrow x, y \text{ relatively prime}$$

- Algorithm: input $\langle x, y \rangle$
  1. Repeat if $y \neq 0$

$$x \leftarrow x \bmod y$$

     exchange x and y
  2. Output x

Euclidean Algorithm II

- The output is the gcd
- Note that in the beginning we don't need

$$x \geq y$$

If

$$x < y,$$

then

$$x = x \bmod y$$

and

$$(x, y) \text{ becomes } (y, x)$$

Euclidean Algorithm III

- Why this works

$$
\begin{aligned}
18 &= ab \\
24 &= ac \\
24 &= 18d + e \\
ac &= abd + e \\
e &= a(c - bd) \\
a &\mid 24 - 18d
\end{aligned}
$$

- Is this algorithm polynomial?
- At each iteration, x or y reduced at least by half

Euclidean Algorithm IV

- If $x > y$

$$x \bmod y \leq x/2$$

Proof

$$\text{if } x/2 \geq y, x \bmod y \leq y \leq x/2$$

$$\text{if } x/2 < y, x \bmod y = x - y \leq x/2$$

- Therefore,

$$\#\text{iterations} \;\; \leq 2\max(\log_2 x, \log_2 y) = O(n)$$

n: length of input (x and y are stored as bit strings), $\log_2 x + \log_2 y$

Euclidean Algorithm V

- Each iteration

  x mod y: polynomial

  see: 1100011 % 101

  $\#$digit $\leq O(n)$: each digit $\leq O(n)$

  exchange x and y: polynomial

Th 7.16 I

- Context-free language $\in P$
- Proof omitted