The use of pumping lemma I

- Recall that from previous examples we need to choose a string s
- Some are confused about why we need to choose s but not p
- Here we will discuss in detail about how we really use the pumping lemma

The use of pumping lemma II

- Let's start from the following statement

$$\forall n, 1 + \cdots + n = \frac{n(n+1)}{2}$$

What is the opposite statement?

$$\exists n \text{ such that } 1 + \cdots + n \neq \frac{n(n+1)}{2}$$

- Formally, in the pumping lemma

$$\text{regular} \Rightarrow \text{some properties}$$

The use of pumping lemma III

"Some properties" are in fact

$$\exists p, \{\forall s \in A, |s| \geq p[\exists x, y, z \text{ such that } s = xyz \text{ and}$$

$$(xy^i z \in A, \forall i \geq 0, \text{ and } |y| > 0 \text{ and } |xy| \leq p)]\}$$

- To do the proof by contradiction, we need the opposite statement of the right-hand side

$$\forall p, \{\exists s \in A, |s| \geq p, [\forall x, y, z \text{ (opposite of}$$
$$(s = xyz \text{ and}$$
$$(xy^i z \in A, \forall i \geq 0, \text{ and } |y| > 0 \text{ and } |xy| \leq p)))]\}$$

The use of pumping lemma IV

- What is the opposite of

$$s = xyz \text{ and } xy^iz \in A, \forall i \geq 0, \text{ and}$$
$$|y| > 0 \text{ and } |xy| \leq p?$$

- Various equivalent forms are possible. The one we consider is

$$(s = xyz \text{ and } |y| > 0 \text{ and } |xy| \leq p)$$
$$\rightarrow \exists i \geq 0, xy^iz \notin A$$

The use of pumping lemma V

- Therefore, the opposite statement of the right-hand side that we really use is

$$
\begin{aligned}
&\forall p, \{\exists s \in A, |s| \geq p, [\forall x, y, z \\
&((s = xyz \text{ and } |y| > 0 \text{ and } |xy| \leq p) \\
&\quad \rightarrow \exists i \geq 0, xy^i z \notin A)]\}
\end{aligned} \tag{1}
$$

- Note that the opposite of

$$A \& B$$

The use of pumping lemma VI

is

$$A \to \neg B$$

See the truth table

A	B	$A \& B$	$\neg(A \& B)$	$\neg B$	$A \to \neg B$
0	0	0	1	1	1
0	1	0	1	0	1
1	0	0	1	1	1
1	1	1	0	0	0

- To prove (1), the "exists" part is important
- You can see that we need to choose s and find an i

The use of pumping lemma VII

- In a sense we guess s or i to see if the statement can be proved.

  If not, we may try other choices
- About

  $$\forall x, y, z, \cdots$$

  in (1) you can see in examples that we go through all possible cases of x, y, z

Example 1.75 I

- $F = \{ww \mid w \in \{0,1\}^*\}$ not regular
- We choose
$$s = 0^p 1 0^p 1 \in F$$
- If
$$s = xyz, |xy| \leq p, |y| > 0,$$
then
$$y = 0 \ldots 0$$
and thus
$$xy^2z = 0 \ldots 010^p1 \neq ww$$

Example 1.75 II

- What if we say there exists $i = 3$ such that

$$xy^iz = xy^3z = 0\ldots010^p1 \neq ww?$$

  The proof is still correct.
- Note that we only need to find an i such that xy^iz is not in the language

Example 1.76 I

- $D = \{1^{n^2} \mid n \geq 0\}$ not regular
- For this language we have

$$
\begin{aligned}
n &= 0, 1^0 = \epsilon \\
n &= 1, 1^1 = 1 \\
n &= 2, 1^4 = 1111 \\
n &= 3, 1^9 = 111111111
\end{aligned}
$$

- We choose

$$s = 1^{p^2} \in D$$

Example 1.76 II

- If
$$s = xyz, |xy| \leq p, |y| > 0$$
then
$$p^2 < |xy^2z| \leq p^2 + p < (p+1)^2$$
and therefore
$$xy^2z \notin D$$

Example 1.76 III

- What if we consider $i = 0$? It seems that

$$(p-1)^2 < p^2 - p \leq |xy^0z| < p^2$$

can also give us xy^0z not in the language. However,

$$(p-1)^2 < p^2 - p$$

may not hold at $p = 1$.
- Note that p is any positive integer now
- However, we still have the proof because as we said, one i is enough