Regular Operations I

- Regular operations can be used to study whether languages are regular or not.
- That is, we aim to check for a given language, whether there are finite automata to recognize it or not.
- Three definitions:
  - $A, B$ are given languages
  - union
    
    $$A \cup B$$
concatenation

\[ A \circ B = \{ xy \mid x \in A, y \in B \} \]

star:

\[ A^* = \{ x_1 \cdots x_k \mid k \geq 0, x_i \in A \} \]
If $k = 0$, what do we mean $x_1 \cdots x_k$?

We define

$\epsilon : \text{empty string}$

in this situation

Thus

$\epsilon \in A^*$
Example

\[ \Sigma = \{a, \ldots, z\} \]
\[ A = \{\text{good, bad}\} \]
\[ B = \{\text{boy, girl}\} \]
\[ A \circ B = \{\text{goodboy, \ldots}\} \]
\[ A^* : \{\epsilon, \text{good, bad, goodgood, \ldots}\} \]

- We say an operation \( R \) is **closed** if the following property holds

  \[ \text{if } x \in A, y \in A, \text{ then } xRy \in A \]
Example: $\mathbb{N} = \{1, 2, \ldots\}$ is closed under multiplication

- **Th 1.25:** regular languages are closed under the union operation

  $A_1, A_2$ are regular languages
  \[\Rightarrow A_1 \cup A_2 \text{ is regular}\]

- **Proof**
Assume we are given two automata

\[ M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \]
\[ M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \]
Construct a new machine

\[ M = (Q, \Sigma, \delta, q, F) \]

\[ Q = \{(r_1, r_2) \mid r_1 \in Q_1, r_2 \in Q_2\} \]

\[ \delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a)) \]

\[ q_0 = (q_1, q_2) \]

\[ F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\} \]

Example: combining

\[ \{w \mid w \text{ has an odd } \# 1's\} \cup \]

\[ \{w \mid w \text{ has an odd } \# 0's\} \]
Regular Operations VIII

- \( q_e \) transitions:
  - 0 to \( q_o \)
  - 1 to \( s_e \)

- \( q_o \) transitions:
  - 0 to \( q_e \)

- \( s_e \) transitions:
  - 1 to \( s_o \)
  - 0 to \( s_e \)

- \( s_o \) transitions:
  - 0 to \( s_e \)
  - 1 to \( q_o \)
Is this proof rigourously enough?
A formal proof should be done by induction. But we don’t provide it here

- Th 1.26: closed under concatenation
  - If $A, B$ are regular, then $A \circ B$ is regular
  - But the proof is not easy
  - It’s unclear where to break the input

- To easily do the proof, we introduce a new technique called **nondeterminism**