Regular Operations I

- Regular operations can be used to study whether languages are regular or not
- That is, these operations can help us to check if for a given language, whether there are finite automata to recognize it or not
- We mainly consider three operations.
- Assume A, B are given languages
  - union

$$A \cup B$$

Regular Operations II

- concatenation

$$A \circ B = \{xy \mid x \in A, y \in B\}$$

- star:

$$A^* = \{x_1 \cdots x_k \mid k \geq 0, x_i \in A\}$$

Regular Operations III

- If $k = 0$, what do we mean

$$x_1 \cdots x_k?$$

We define

$$\epsilon : \text{ empty string}$$

in this situation

- Thus

$$\epsilon \in A^*$$

Regular Operations IV

- Example

$$\begin{aligned} \Sigma &= \{a, \ldots, z\} \\ A &= \{good, bad\} \\ B &= \{boy, girl\} \\ A \circ B &= \{goodboy, \ldots\} \\ A^* &: \{\epsilon, good, bad, goodgood, \ldots\} \end{aligned}$$

- We say an operation R is closed if the following property holds

$$\text{if } x \in A, y \in A, \text{ then } xRy \in A$$

Regular Operations V

Example: $N = \{1, 2, \ldots\}$ is closed under multiplication

- Th 1.25: regular languages are closed under the union operation

$$\begin{aligned} & A_1, A_2 \text{ are regular languages} \\ \Rightarrow & A_1 \cup A_2 \text{ is regular} \end{aligned}$$

- Proof

Regular Operations VI

Assume we are given two automata

$$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$$
$$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$$

- Question: you want to think about why we can consider the same Σ
- Idea: we construct a parallel machine to run two machines simultaneously

Regular Operations VII

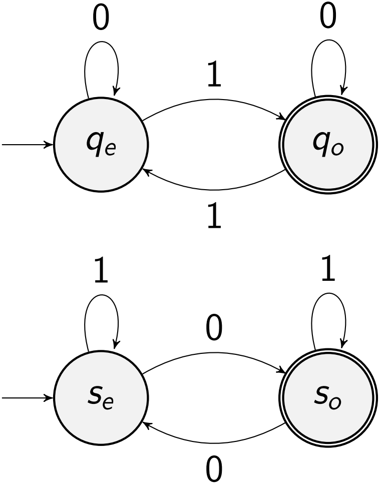
- Definition of our new machine

$$\begin{aligned} M &= (Q, \Sigma, \delta, q_0, F) \\ Q &= \{(r_1, r_2) \mid r_1 \in Q_1, r_2 \in Q_2\} \\ \delta((r_1, r_2), a) &= (\delta_1(r_1, a), \delta_2(r_2, a)) \\ q_0 &= (q_1, q_2) \\ F &= \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\} \end{aligned}$$

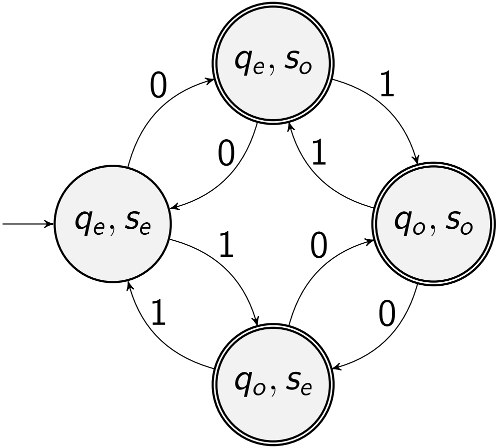
- Example: combining

$$\{w \mid w \text{ has an odd } \# \text{ 1's}\} \cup$$
$$\{w \mid w \text{ has an odd } \# \text{ 0's}\}$$

Regular Operations VIII



Regular Operations IX



- Is this proof rigorously enough?

Regular Operations X

A formal proof should be done by induction. But we don't provide it here

- Th 1.26: closed under concatenation

  $$\text{If } A, B \text{ are regular, then } A \circ B \text{ is regular}$$

  But the proof is not easy

  It's unclear where to break the input
- To easily do the proof, we introduce a new technique called nondeterminism