Mathematical notions I

- Set
  Omitted
- Sequence and tuples
  - Sequence: Objects in order

$$(7, 21, 57) \neq (57, 7, 21)$$

  - Repetition

$$\begin{aligned} \text{set} &: \{7, 21, 57\} = \{7, 7, 21, 57\} \\ \text{sequences} &: (7, 21, 57) \neq (7, 7, 21, 57) \end{aligned}$$

Mathematical notions II

- Tuples: finite sequence
  (7,21,57): 3-tuple
- Cartesian product:

$$A = \{1, 2\}, B = \{x, y\}$$
$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

- Function: single output
- Relation: scissors-paper-stone

beats	scissors	paper	stone
scissors	F	T	F
paper	F	F	T
stone	T	F	F

Mathematical notions III

- Equivalence relation
  1. reflexive
  $$\forall x, xRx$$
  2. symmetric
  $$xRy \Leftrightarrow yRx$$
  3. transitive
  $$xRy, yRz \Rightarrow xRz$$

  e.g. "$=$"

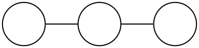
Mathematical notions IV

- Example: $i \equiv_7 j$ if $0 = i - j \mod 7$

$$\begin{aligned} & i - i \mod 7 = 0 \\ & i - j = 7a, j - i = -7a \\ & i - j = 7a, j - k = 7b \\ & \Rightarrow i - k = 7(a + b) \end{aligned}$$

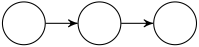
- Graph

  Undirected



  Directed

Mathematical notions V



Nodes (vertices)

- Edges: connection between nodes

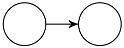
Degree = # edges at a node

Subgraph: G is subgraph of H if

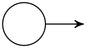
  - G is a graph
  - node(G) $\subset$ node(H)
  - edge(G) = subset of edge(H) connecting node(G)

In our example,

Mathematical notions VI



is a subgraph, but



is not

- Strings and languages
  - alphabet: $\{0, 1\}$
  - string: 1001
  - language: set of strings
- Boolean logic
  - true and false

Mathematical notions VII

- 0 (false) and 1 (true)
- $0 \wedge 0 = 0, 0 \vee 0 = 0, \neg 0 = 1$ (negation operation)
- xor $\otimes$

$$
\begin{aligned}
0 \otimes 0 &= 0 \\
0 \otimes 1 &= 1 \\
1 \otimes 0 &= 1 \\
1 \otimes 1 &= 0
\end{aligned}
$$

- implication

Mathematical notions VIII

P	Q	$P \to Q$
0	0	1
0	1	1
1	0	0
1	1	1

The above is called a truth table

- Why

$$P = 0, Q = 1, \text{ then } P \to Q = 1?$$

Consider

$$\text{rainy} \to \text{wet land}$$

Mathematical notions IX

If not rainy, saying rainy implies wet land is ok.

- $P \to Q \equiv \neg P \vee Q$

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Proof I

- Direct proof:

$$A \to B$$

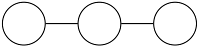
- Proof by contradiction

$$\neg B \to \neg A$$

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Proof II

- Example 1:
  Every graph $\Rightarrow$ sum of degrees is even
  - An example:



$$\# \text{ degrees} = 1 + 2 + 1 = 4$$

  - Each edge: 2 nodes

$$\text{total } \# \text{ degrees} = 2 \times \ \# \text{ edges}$$

  What is the left side of the implication? It's the definition of graphs
- Example 2: $\sqrt{2}$ is irrational

Proof III

- The implication

$$\begin{aligned}&\text{Definition of rational numbers}\\ \Rightarrow&\sqrt{2} \text{ is not rational}\end{aligned}$$

That is,

$$\begin{aligned}&\text{If a rational number is ...}\\ \Rightarrow&\sqrt{2} \text{ is not rational}\end{aligned}$$

The opposite is

$$\begin{aligned}&\text{If } \sqrt{2} \text{ is rational}\\ \Rightarrow&\text{The rational number cannot be defined as ...}\end{aligned}$$

Proof IV

- By definition, $\sqrt{2}$ is rational means that

$$\sqrt{2} = \frac{m}{n}$$

  and m, n have no common factor
- Then

$$2n^2 = m^2$$

  Looks impossible. But how to write this formally?
- First we prove that m must be even. This is also proof by contradiction

Proof V

If m is not even,

$$m = 2k + 1.$$

Then

$$m^2 = 4(k^2 + k) + 1$$

is not even and

$$m^2 = 2n^2$$

does not hold.

Proof VI

- Now suppose m is even

$$m = 2k$$

Then

$$n^2 = 2k^2$$

- By the same argument, n is even
- Thus m, n have a common factor 2 and there is a contradiction