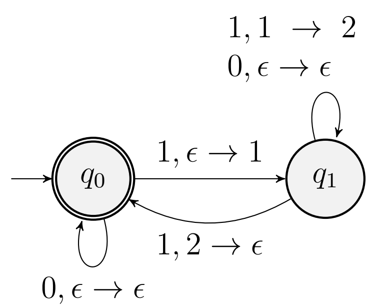
Introduction to the Theory of Computation 2021 — Final Exam

Solutions

Problem 1 (5 pts). Consider the following language

$$\{w \in \{0,1\}^* | \text{ sum of } w \bmod 3 = 0\}, \tag{1}$$

where ϵ is also included in the language. Recall in the second exam, for the case of mod $6 = 0$, we have a 2-state PDA for the language. Now for (1) the 2-state PDA is



Can this diagram be considered as a DPDA? If so, give the full table of δ. Otherwise, give the reasons why it is not a DPDA.

Solution.

Yes, the diagram above is a DPDA since for each state, input and top of stack, there is always only one possible transition (i.e. $\forall q \in Q, a \in \Sigma, x \in \Gamma$, exactly one of $\delta(q,a,x), \delta(q,a,\epsilon), \delta(q,\epsilon,x), \delta(q,\epsilon,\epsilon)$ is not $\emptyset$). The table of δ is:

input	ϵ	0			1		
stack	$\epsilon, 1, 2$	ϵ	1	2	ϵ	1	2
q_0	$\emptyset$	(q_0, ϵ)	$\emptyset$	$\emptyset$	$(q_1, 1)$	$\emptyset$	$\emptyset$
q_1	$\emptyset$	(q_1, ϵ)	$\emptyset$	$\emptyset$	$\emptyset$	$(q_1, 2)$	(q_0, ϵ)

Comment mistakes: Your Γ does not need to have 0.

Problem 2 (25 pts). For the language (1), consider the standard TM (i.e., one-tape deterministic TM) to decide it. For subproblems (a) and (b), you are restricted to use

$$\Sigma = \{0,1\} \text{ and } \Gamma = \Sigma \cup \{\sqcup\}.$$

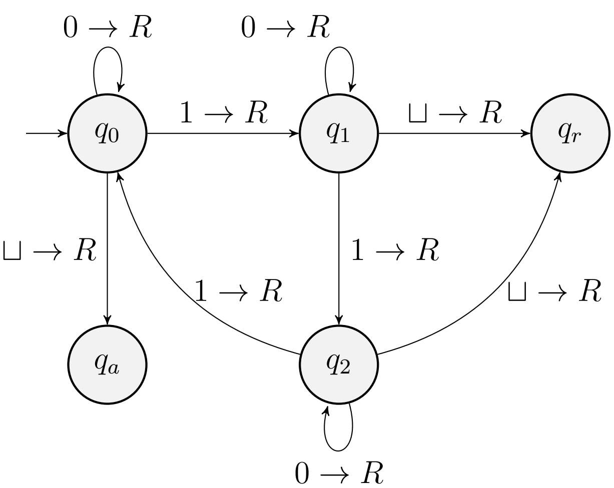
(a) (10 pts) Give the state diagram and the formal definition of a TM that decides language (1). The total number of states (i.e. $|Q|$, including q_{accept} and q_{reject}) should be less than or equal to 5. All states including q_{reject} should be drawn in the diagram.

(b) (10 pts) Give the state diagram of a two-tape TM that decides language (1) with only three states. Note that following the textbook, we allow S for the head movement.

(c) (5 pts) Simulate your machine in (b) on the input 1011. Show the configuration of the machine in each step.

Solution.

(a) We only need to simulate the DFA we already know using a TM. So, the state diagram can be



The formal definition is

$Q = \{q_0, q_1, q_2, q_a, q_r\}$

$\Sigma = \{0, 1\}$

$\Gamma = \{\sqcup, 0, 1\}$

Start state: q_0

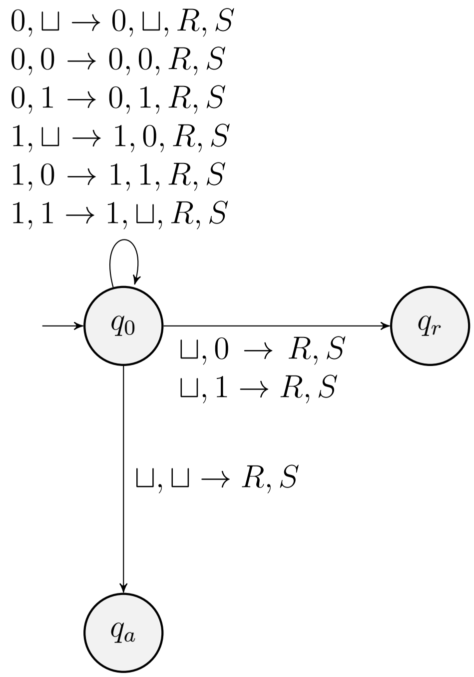
Accept state: q_a

Reject state: q_r

δ :

	0	1	$\sqcup$
q_0	(q_0, R)	(q_1, R)	(q_a, R)
q_1	(q_1, R)	(q_2, R)	(q_r, R)
q_2	(q_2, R)	(q_0, R)	(q_r, R)

(b) The idea is pretty similar to the PDA in problem 1. The machine reads the input tape just like a DFA (always going right). We use the second tape just like how we used the stack. Although we do not have symbol 2, we have $0, 1$ and $\sqcup$ that allow us to represent three different states on the second tape. We use $\sqcup$ to represent the state that the sum mod 3 is 0, 0 for the case which sum is 1 and 1 for the case sum is 2. The diagram can be



(c)

$$\begin{matrix} q_0 1011\sqcup \\ q_0\sqcup \end{matrix} \Longrightarrow \begin{matrix} 1q_0 011\sqcup \\ q_0 0 \end{matrix} \Longrightarrow \begin{matrix} 10q_0 11\sqcup \\ q_0 0 \end{matrix} \Longrightarrow \begin{matrix} 101q_0 1\sqcup \\ q_0 1 \end{matrix} \Longrightarrow \begin{matrix} 1011q_0\sqcup \\ q_0\sqcup \end{matrix} \Longrightarrow \text{accept}$$

Problem 3 (20 pts). Consider

$$f(n) = \frac{2n + 80}{n^3}.$$

(a) (10 pts) Prove

$$f(n) = O(\frac{1}{n^2})$$

by using the definition of big-O. That is, you must give some c and n_0 so the definition holds.

(b) (10 pts) Prove

$$f(n) = o(\frac{1}{n})$$

by using the definition of the limit, i.e., you cannot directly calculate the limit. Furthermore, for any given c, please determine the smallest n_0.

Solution.

(a) Our target is finding $c > 0$ and $n_0 > 0$ such that

$$\begin{aligned} &\frac{2n + 80}{n^3} \le c\frac{1}{n^2} \\ \Rightarrow &c\frac{1}{n^2} - \frac{2n + 80}{n^3} \ge 0 \\ \Rightarrow &\frac{cn - 2n - 80}{n^3} \ge 0 \\ \Rightarrow &\frac{(c - 2)n - 80}{n^3} \ge 0 \end{aligned}$$

for all $n > n_0$. Thus, we can take

$$c = 3$$

and

$$n_0 = 80,$$

so that

$$\frac{(c-2)n - 80}{n^3} = \frac{n - 80}{n^3}$$

is always greater than or equal to zero as $n > n_0 = 80$.

(b) The definition of the limit is

$$\forall c > 0, \exists n_0, \forall n \geq n_0, \frac{2n + 80}{n^3} \leq c\frac{1}{n},$$

which is equivalent to

$$\begin{aligned} & c\frac{1}{n} - \frac{2n+80}{n^3} \geq 0 \\ \Rightarrow & \frac{cn^2 - 2n - 80}{n^3} \geq 0 \\ \Rightarrow & \frac{c(n - 1/c)^2 - 80 - 1/c}{n^3} \geq 0. \end{aligned}$$

We also know that the largest root of

$$h(n) = c(n - \frac{1}{c})^2 - 80 - \frac{1}{c}$$

is

$$\tilde{n} = \frac{\sqrt{80c + 1} + 1}{c}$$

and $h(n)$ is always greater than zero as $n > \tilde{n}$. Therefore, the smallest n_0 is equal to $\lceil \tilde{n} \rceil$, so that

$$\frac{c(n - 1/c)^2 - 80 - 1/c}{n^3} \geq 0$$

for all $n \geq n_0 = \lceil \tilde{n} \rceil$.

We can also list the n_0 (calculated above) corresponding to each c in a table:

c	n_0
1	10
2	7
3	6
$[4, 5]$	5
$[6, 9]$	4
$[10, 20]$	3
$[21, 81]$	2
$[82, \infty]$	1

Problem 4 (15 pts). Consider the following language

$$\{w\#w\#\cdots\#w \mid w \in \{0,1\}^*\},$$

which includes ϵ.

(a) (10 pts) Give the diagram of a 2-tape TM (S is allowed) to recognize this language with ≤ 6 states (including q_{accept},q_{reject}) and

$$\Gamma = \{0, 1, \sqcup, \#\}$$

by the following concepts

i) Copy w to the tape 2.

ii) If the machine reads # in the tape 1, it moves the head of the tape 2 to the beginning.

iii) Continuously move the right for comparing contents in the tape 1 and the tape 2. If the machine reads # and $\sqcup$ in the tape 1 and 2 respectively, go to ii). If the machine reads $\sqcup$ and $\sqcup$ in the tape 1 and 2 respectively, it goes to the accept state.

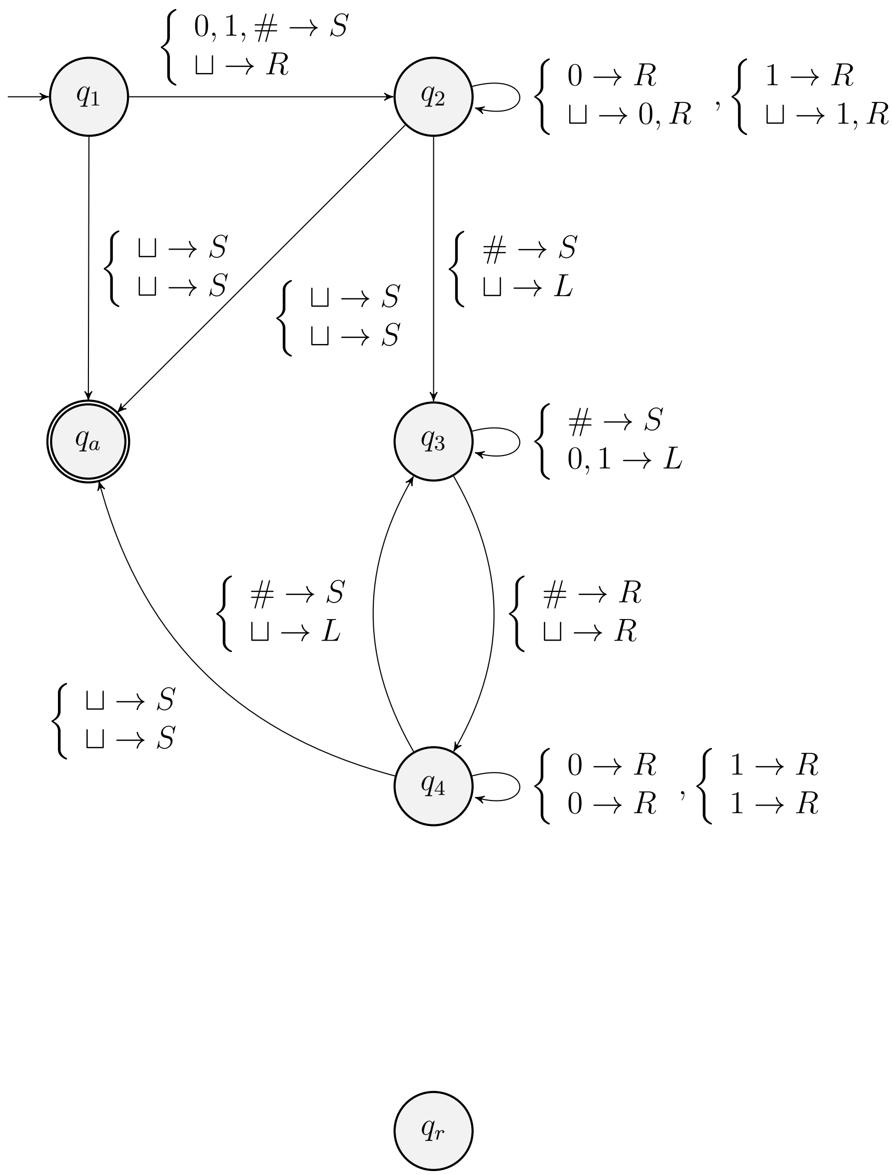
The paths connected to q_{reject} can be omitted.

(b) (5 pts) Simulate the following input string

$$00\#00.$$

Solution.

(a) The diagram can be designed as



,

where we do not show the paths connected to q_r. From q_1 to q_2, we have a $\sqcup$ in tape 2 to indicate the beginning position. From q_2 to q_2, we copy w to tape 2. From q_2 to q_3 and q_3 to q_3, we hold the head of the tape 1 on the # position and move the head of the tape 2 to the beginning position. From q_3 to q_4 and q_4 to q_4, we compare the contents in tape 1 and tape 2. If the machine reads # in tape 1 and the contents in tape 1 and tape 2 are the same, we will go back to q_3 from q_4. If the machine reads $\sqcup$ in both tape 1 and tape 2, and the contents in tape 1 and tape 2 are the same, the machine goes to q_a from q_4. Moreover, we handle the empty word ϵ from q_1 to q_a, and handle the word w without # from q_2 to q_a.

(b) Let us simulate 00#00 as

$$
\begin{array}{lll}
\begin{array}{ccccccc} q_1 & 0 & 0 & \# & 0 & 0 & \sqcup \\ q_1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \end{array}
& \Rightarrow \begin{array}{ccccccc} q_2 & 0 & 0 & \# & 0 & 0 & \sqcup \\ \sqcup & q_2 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \end{array}
& \Rightarrow \begin{array}{ccccccc} 0 & q_2 & 0 & \# & 0 & 0 & \sqcup \\ \sqcup & 0 & q_2 & \sqcup & \sqcup & \sqcup & \sqcup \end{array} \\
\Rightarrow \begin{array}{ccccccc} 0 & 0 & q_2 & \# & 0 & 0 & \sqcup \\ \sqcup & 0 & 0 & q_2 & \sqcup & \sqcup & \sqcup \end{array}
& \Rightarrow \begin{array}{ccccccc} 0 & 0 & q_3 & \# & 0 & 0 & \sqcup \\ \sqcup & 0 & q_3 & 0 & \sqcup & \sqcup & \sqcup \end{array}
& \Rightarrow \begin{array}{ccccccc} 0 & 0 & q_3 & \# & 0 & 0 & \sqcup \\ \sqcup & q_3 & 0 & 0 & \sqcup & \sqcup & \sqcup \end{array} \\
\Rightarrow \begin{array}{ccccccc} 0 & 0 & q_3 & \# & 0 & 0 & \sqcup \\ q_3 & \sqcup & 0 & 0 & \sqcup & \sqcup & \sqcup \end{array}
& \Rightarrow \begin{array}{ccccccc} 0 & 0 & \# & q_4 & 0 & 0 & \sqcup \\ \sqcup & q_4 & 0 & 0 & \sqcup & \sqcup & \sqcup \end{array}
& \Rightarrow \begin{array}{ccccccc} 0 & 0 & \# & 0 & q_4 & 0 & \sqcup \\ \sqcup & 0 & q_4 & 0 & \sqcup & \sqcup & \sqcup \end{array} \\
\Rightarrow \begin{array}{ccccccc} 0 & 0 & \# & 0 & 0 & q_4 & \sqcup \\ \sqcup & 0 & 0 & q_4 & \sqcup & \sqcup & \sqcup \end{array}
& \Rightarrow \begin{array}{ccccccc} 0 & 0 & \# & 0 & 0 & q_a & \sqcup \\ \sqcup & 0 & 0 & q_a & \sqcup & \sqcup & \sqcup \end{array} &
\end{array}
$$

Problem 5 (10 pts). Consider

$$
\begin{aligned}
f(n) &= n^3 \\
g(n) &= 2^n
\end{aligned}
$$

(a) (5 pts) Is

$$f(n) \times g(n) = O(2^n)?$$

(b) (5 pts) Is

$$f(n) \times g(n) = 2^{O(n)}?$$

You cannot just answer yes or no. To have "=", you need to prove the definition of big-O. To have "$\neq$" you must prove the opposite statement of the big-O definition.

Solution.

(a) No. Let us assume that

$$f(n) \times g(n) = O(2^n).$$

By the definition, there exists $c > 0$ and n_0 such that

$$
\begin{aligned}
&c2^n - n^3 2^n \geq 0 \\
\Rightarrow &(c - n^3)2^n \geq 0
\end{aligned}
$$

as $n > n_0$. However, for any $c > 0$ and any n_0, we consider

$$n > \max(n_0, \sqrt[3]{c})$$

such that

$$(c - n_0^3)2^{n_0} < 0$$

Thus,

$$f(n) \times g(n) \neq O(2^n).$$

Comment mistakes: The opposite statement has that for any c, n_0, we choose n, so you cannot choose n_0.

(b) Yes. We have

$$n^3 < 2^n$$

when $n > 10$. Thus,

$$f(n) \times g(n) = n^3 2^n < 2^n \cdot 2^n = 2^{2n}$$

for all $n > 10$. By taking

$$c = 2$$

and

$$n_0 = 10,$$

we have proved that

$$f(n) \times g(n) = 2^{O(n)}$$

by the definition.

Problem 6 (25 pts). In problem 2(a), we used five states to construct the TM. We would like to prove that a four-state standard TM is not possible. However, the problem seems to be difficult, so we decide to simplify the language in (1) to

$$L_3 = \{w \in \{1\}^* | \text{ sum of } w \bmod 3 = 0\},$$

where $\epsilon \in L_3$ and prove that no four-state TM can **decide** it. We give the proof below, but **three parts** are taken out. Your task is to fill those parts.

For the proof, we (and you) may use the following lemmas. Due to the interest of time, you can skip reading their proofs.

Lemma 1. *Let M be a TM that recognizes L_3 and w be a string that is accepted or rejected by M (including rejection by an infinity loop). Then, before accepting or rejecting w, M must have read the $\sqcup$ at the end of the input. Moreover, M takes at least $|w|$ steps before accepting or rejecting w.*

Proof of Lemma 1. Suppose that M is a TM that recognizes L_3, and M accepts or rejects some string w without ever reading the $\sqcup$ at the end of the input. Then, M must make the same decision of w for the strings $w1$ and $w11$. As a result, M does not recognize L_3 because it is not possible that w, $w1$ and $w11$ are all in or not in L_3. Therefore, M must have read the $\sqcup$ at some point. Further, to read up to this $\sqcup$ at the end of the input, M needs at least $|w|$ steps to move the head there. □

Lemma 2. *There does not exist a TM with*

$$|Q| = 3, \Sigma = \{1\}, \Gamma = \{1, \sqcup\}$$

that recognizes L_3.

Proof of Lemma 2. Suppose there exists a machine M that recognize L_3 with

$$Q = \{q_0, q_a, q_r\},$$

where q_0, q_a and q_r are the start, accept and reject states, respectively. A TM must terminate when it goes to q_a or q_r. Thus, there are no transitions starting from q_a and q_r, so we only need to discuss the transitions from q_0 to q_a, q_0 to q_r and q_0 to q_0. Furthermore, because

$$|\Gamma| = 2,$$

we know that

$$\begin{aligned}&\text{there are exactly two transitions from } q_0 \text{ to some states}\\&\text{by respectively handling input characters 1 and } \sqcup .\end{aligned} \tag{2}$$

To accept the strings in L_3, there must be a transition

$$q_0 \xrightarrow{x} q_a, \tag{3}$$

where $x \in \Gamma$. If x is 1, M would accept 1, which is a contradiction. Therefore, (3) must be

$$q_0 \xrightarrow{\sqcup} q_a. \tag{4}$$

By (2) and (4), we only need to discuss two cases for the other transition:

Case 1:

$$q_0 \xrightarrow{1} q_r$$

Case 2:

$$q_0 \xrightarrow{1 \to ?, A} q_0 \tag{5}$$

where $A \in \{R, L\}$.

For Case 1, M rejects 111, which is a contradiction. For Case 2, if A is L, then M can only read the first cell of the tape. Because M recognizes L_3, M would accept 111 without ever touching the $\sqcup$ at the end of input, which is a contradiction by Lemma 1. Therefore, A should be R. For the input string 1, after one transition by (5), the head points to a $\sqcup$, so M takes (4) to accept the string. Then we get a contradiction. □

Now, we can begin to prove there are no four-state TM that decides L_3. Suppose there is a machine M that decides L_3 with

$$\Gamma = \{1, \sqcup\}, Q = \{q_0, q_1, q_a, q_r\},$$

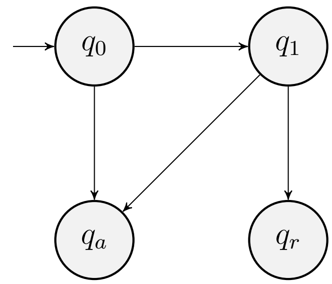
where q_0, q_a and q_r are the start, accept and reject states, respectively. Because there are no transitions starting from q_a and q_r, we only have to consider 4 transitions in total (two of the transitions start from q_0, the other two start from q_1). Hence, we have the following discussion:

Discussion 1: The first transition must be a transition $? \to q_r$. Because M is a decider, M does not rely on infinite loops to reject a string. Therefore, M must reject some string by transitioning to q_r.

Discussion 2: The second transition must be $? \to q_a$, since M must accept some strings.

Discussion 3: The third transition must be $q_0 \to q_1$. If such a transition does not exist, q_1 is never reached and therefore M is equivalent to a three-state TM. By Lemma 2, M cannot decide L_3, so there is a contradiction.

Discussion 4: Following Discussion 1-3, the last transition can neither go to q_a nor q_r. Otherwise, the machine can only transition at most once before deciding. For example, assume that the diagram is like



Then, after one step (q_0 to q_1), the TM terminates by going to q_a or q_r. By the assumption that M decides L_3, Lemma 1 implies that M accepts 111 with at least three steps. Thus, there is a contradiction.

From Discussion 1-4, we know that there are exactly one transition to q_a and exactly one transition to q_r. We can then separate the discussion into two cases:

(a) Both transitions to q_a and q_r read $\sqcup$

Because the language includes ϵ, the machine should have

$$q_0 \xrightarrow{\sqcup} q_a \tag{6}$$

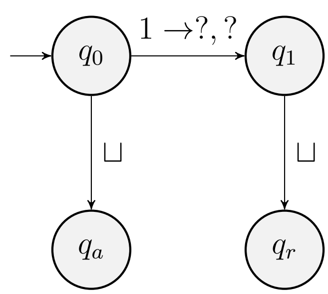
instead of

$$q_0 \xrightarrow{\sqcup} q_r.$$

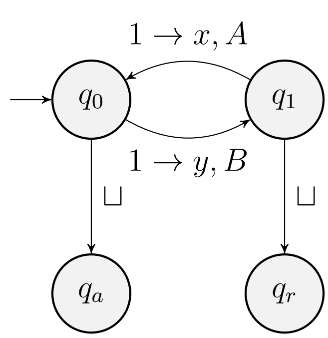
Since the transition to q_r also reads $\sqcup$, the determinism of TM and (6) imply that a transition must be

$$q_1 \xrightarrow{\sqcup} q_r.$$

Also, by Discussion 3, there must be a transition from q_0 to q_1. Now, the diagram looks like:



Next, when reading 1 in q_1, it should go back to q_0. Otherwise, M either loops at q_1 or goes to q_r. In either case, M does not accept 111, which is a contradiction. Thus, the diagram should look like



where $x, y \in \Gamma$ and $A, B \in \{L, R\}$.

(10 pts) **Please finish the proof of this case by checking all possibilities of A and B.** Then, conclude that all of them lead to contradiction. (Hint: Using the lemmas properly should make things easier)

(b) At least one of the two transitions to q_a and q_r reads 1.

By the fact that one of the two deciding transitions reads 1 and M is a decider, there must be a string w that is decided when reading 1. For this w, Lemma 1 implies that M must have read the $\sqcup$ at the end of input before acceptance or rejection. If after reading this $\sqcup$, the head of M only goes right (i.e., all subsequent transitions are $? \to ?, R$), M does not have a chance to read a 1 afterward. However, we already knew that for this w, the last transition before reaching q_a or q_r reads a 1. Therefore, we conclude that

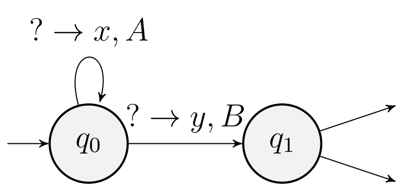
$$\text{“for the two transitions going to neither } q_a \text{ nor } q_r\text{, one must move left.”} \tag{7}$$

From Discussion 1-4, there are four transitions in total. That is, $q_0 \to q_1$, $? \to q_a$, $? \to q_r$ and the last one goes to neither q_a nor q_r. Therefore, there are only 4 possibilities of the last edge:

(i) $q_0 \to q_0$
(ii) $q_0 \to q_1$
(iii) $q_1 \to q_0$
(iv) $q_1 \to q_1$

We discuss each of them separately. Note that in each corresponding diagram, there are two transitions without showing the target states (e.g., two from q_1 to nowhere in the first figure). They are the transitions to q_a and q_r. In all cases, the assignment of the target state q_a or q_r does not affect our discussion, so they do not need to be specified.

(i)



According to (7), at least one of A and B must be L. We separately consider the following cases:

Case 1. $A = B = L$: M can not read more than one input character since it does not have any transition that moves right. By the assumptions that M recognizes L_3, M accepts 111 without ever reading the $\sqcup$ after of input. This contradicts Lemma 1.

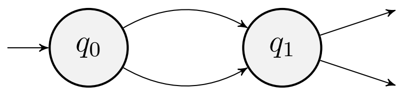
Case 2. $A = L$ and $B = R$: In this case, when M reaches q_1 before deciding, at most two input characters have been read. By the same reason as in case b(i)1, a contradiction occurs by Lemma 1.

Case 3. $A = R$ and $B = L$: Let w be any input string such that $|w| \geq 3$. If M does not use the transition $q_0 \to q_0$, it must decide w in only two steps, which contradicts Lemma 1 since $|w| \geq 3$. Because $A = R, B = L$ and that $q_0 \to q_0$ is used at least once, the sequence of transitions must be:

$$q_0 \to q_0 \to \cdots \to q_0 \xrightarrow{? \to x, R} q_0 \xrightarrow{? \to y, L} q_1 \longrightarrow$$

This implies that either M rejects both of 111 and 1111, or accepts both of 111 and 1111. Thus, there is a contradiction.

(ii)



This machine makes decision after reading at most the first two tape cells. Therefore, it must accept 111 before reading the ⊔ next to the input. This contradicts Lemma 1.

(iii) (5 pts) **Please finish the proof for the case where the last edge is** $q_1 \to q_0$.

(iv) (10 pts) **Please finish the proof for the case where the last edge is** $q_1 \to q_1$.

All of the four cases lead to contradictions. Thus, there are no 4-state TM that decides L_3 in this case.

Solution.

(a) We can list all possibilities of A and B and give a reason for their contradictions in the following table:

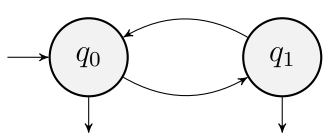
A, B	contradiction
$A = B = L$	can only read the first input
$A = L, B = R$	can only read the first two input
$A = R, B = L$	can only read the first two input
$A = B = R$	no matter of what x, y is, it accepts 11

Because the fourth case in the table cannot happen, M satisfies one of the first three cases in the table, which indicates that M only reads the first or two cells. By the assumption that M decides L_3, M accepts 111 without reading the ⊔ at the end of 111. This is a contradiction to Lemma 1. Therefore, no 4-state standard TM can decide L_3 in this case.

Common mistake: Some directly wrote: If $A = L, B = R$, then 11 cause an infinite loop. You cannot have such result without discussing x and y. Indeed, if $y = ⊔$, then 11 is accepted.

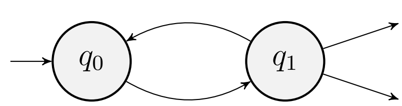
(b)

(iii)



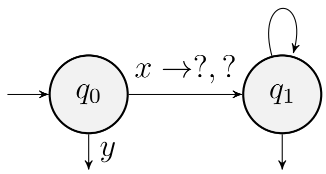
From (7), one of the two transitions between q_0 and q_1 must move left. Then, the machine can not read more than two tape cells. By the same explanation as in the case (bii), this situation is not possible.

Common mistake: If your figure is wrong, then it is impossible to have a correct proof. For example, some have:



However, at q_1 you should only have two transitions going out.

(iv)



Due to the determinism of TM, in this diagram we have

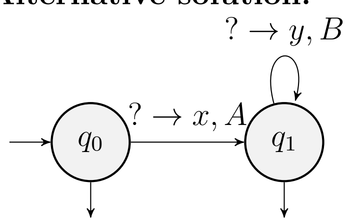
$$\{x, y\} = \{1, \sqcup\} \text{ and } x \neq y.$$

Then, we consider the following possible situations:

Case 1. $x = 1, y = \sqcup$: Because M is a decider, it must not loop forever on any input. Therefore, M make the same decision for all input strings that start with 1. That is, either 111 and 11 are both rejected, or the machine accepts both 111 and 11. This situation is thus impossible.

Case 2. $x = \sqcup, y = 1$: The machine immediately goes to q_a or q_r for any input that starts with 1. Thus, it makes the same decision for all input that starts with 1. By the same reason as the above case, this situation is not possible.

Alternative solution:



By (7), at least one of A and B is L. Then, we can discuss three cases:

i. If $A = B = L$, the machine only read the first tape cell. By the assumption that M decides L_3, M accepts 111 without reading $\sqcup$ at the end of input. We have a contradiciton to Lemma 1.

ii. If $A = R$ and $B = L$, the machine only read the first two tape cells. By the same reasons as in b(iv)i, we have a contradiction.

iii. Suppose $A = L$ and $B = R$. Consider any input string w such that $|w| \geq 2$. If $q_1 \to q_1$ is not used when deciding w, M can not read the $\sqcup$ at the end of input, which contradict Lemma 1. By the fact that M is a decider and $q_1 \to q_1$ must be used when processing long inputs, M would always transition first to q_1 and end up in the same final state (since there is only one transition out of q_1). This implies M either accepts both of 111 and 1111, or reject both of them, a contradiction.