In Chapter 3 we showed that various TMs are equivalent.

For example, single-tape and multi-tape TMs are equivalent.

However, their “time complexity” are different.
Theorem 7.8

$O(t(n))$ multi-tape TM

$\Rightarrow \exists$ equivalent $O(t(n)^2)$ single-tape TM

Idea for the proof: similar to how we proved their equivalence

Show that simulating each step of a multi-tape TM takes $O(t(n))$ on a single-tape TM

Let $k$ be the number of tapes

How did we simulate a multi-tape TM?
Complexity of Multi-tape TM II

- Tape 1: 0 1 1 0 ...
- Tape 2: a a c a ...
- Tape 3: 0 0 1 1 ...
- CPU Actions:
  - # 0 i 1 ...
  - # à a ...
  - # 0 0 …
Simulate each step of multi-tape TM: scan to know where heads point to and do the update

However, we may have to right shift the tape

So we need to know the tape length. It is

\[ k \times O(t(n)) = O(t(n)) \]

Why \( O(t(n)) \)?

Each tape of multi-tape TM has \( O(t(n)) \) length

An \( O(t(n)) \) multi-tape TM generates \( O(t(n)) \) contents in \( O(t(n)) \) time (note that \( O(t(n)) \geq n \)).
Thus the cost of simulating each step of multi-tape TM on a single-tape TM is $O(t(n))$.

There are $O(t(n))$ multi-tape TM steps, so the total cost is

$$O(t(n)) \times O(t(n)) = O(t(n)^2)$$
Remember NTM a decider if all branches halt on all inputs

Definition of NTM time complexity $f(n)$: maximum number of steps the machine uses on any branch on any input length $n$

Theorem 7.11

For an $t(n) \geq n, O(t(n))$ NTM (single tape)

$\Rightarrow \exists$ a $2^{O(t(n))}$ TM (single tape)

Assume $b$ is the maximal number of branches at each layer
Recall that we use a depth-first way for the simulation.

Total number of nodes in the tree:

\[ O(b^{t(n)}) \]

- Depth \( d \) finished, do \( d + 1 \)
- Tape 3: all possible paths so far
- Tape 2: original input \( w \), simulate one path, 1 step further with all possible branches
Cost of running from root to one node in tape 2: $O(t(n))$

Update of tape 3: $O(b^{t(n)})$

Total cost $\leq O(b^{t(n)})$ for each node

Total time:

$$\# \text{ nodes} \times \text{cost per node} = O(b^{t(n)}) \times O(b^{t(n)}) = O((b^2)^{t(n)}) = O(2^{t(n)})$$
This is by a 3-tape TM

To use a single-tape TM to simulate a 3-tape one, we need

\[
(2^{O(t(n))})^2 = 2^{O(t(n))}
\]

cost because

\[
(2^{O(t(n))})^2 \leq (2^{ct(n)})^2 = 2^{2ct(n)} = 2^{O(t(n))}
\]