Some languages not Turing-recognizable

- \( \Sigma^* \) is countable
  Simply count \( w \) with \( |w| = 0, 1, 2, 3, \ldots \)
  For example, if \( \Sigma = \{0, 1\} \), then
  \[ \{\epsilon, 0, 1, 00, 01, 10, 11, \ldots\} \]

- The set of TMs is countable
  Each machine can be represented as a finite string
  (think about the formal definition)
  Thus the set of TMs is a subset of \( \{0, 1\}^* \)
  Let
Some languages not Turing-recognizable II

$L$: all languages over $\Sigma$
$B$: all infinite binary sequences

- For any
  
  $A \in L$

  there is a corresponding element in $B$

- Example:

  $A : 0\{0, 1\}^*$
  
  $\Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$
  
  $A = \{0, 00, 01, 000, 001, \ldots\}$
  
  $\chi_A = 010110011\ldots$
Some languages not Turing-recognizable

- One-to-one correspondence between \( B \) and \( L \)
- \( B \) is uncountable (like real numbers)
  - Therefore, \( L \) is uncountable
- Each TM \( \Rightarrow \) handles one language in \( L \)
  - Set of TM is countable, but \( L \) is not
- Thus some languages cannot be handled by TM
Recall the halting problem is

\[ A_{TM} = \{ \langle M, w \rangle \mid M : TM, \text{accepts } w \} \]

We prove it is undecidable by contradiction.

Assume there is an \( H \) that is a decider for \( A_{TM} \)

Then \( H \) satisfies

\[ H(\langle M, w \rangle) = \begin{cases} 
\text{accept} & \text{if } M \text{ accepts } w \\
\text{reject} & \text{otherwise}
\end{cases} \]

Construct a new TM \( D \) with \( H \) as a subroutine.
For $D$, the input is $\langle M \rangle$, where $M$ is a TM. It runs $H$ on $\langle M, \langle M \rangle \rangle$ and outputs the opposite result of $H$.

The machine $D$ satisfies

$$D(\langle M \rangle) = \begin{cases} 
\text{accept} & \text{if } M \text{ rejects } \langle M \rangle \\
\text{reject} & \text{if } M \text{ accepts } \langle M \rangle
\end{cases}$$

But we get a contradiction:

$$D(\langle D \rangle) = \begin{cases} 
\text{accept} & \text{if } D \text{ rejects } \langle D \rangle \\
\text{reject} & \text{if } D \text{ accepts } \langle D \rangle
\end{cases}$$
We said earlier that the diagonalization method is used for the proof. Is that the case?

We show that indeed it is used.
**Diagonalization in the proof I**

- Set of TMs is countable so we can have

<table>
<thead>
<tr>
<th></th>
<th>$\langle M_1 \rangle$</th>
<th>$\langle M_2 \rangle$</th>
<th>$\langle M_3 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>$M_2$</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>$M_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
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</tbody>
</table>

blank entries: unknown if reject or loop

- But $H$ knows the solution as it is a decider

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<thead>
<tr>
<th></th>
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<th>$\langle M_2 \rangle$</th>
<th>$\langle M_3 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>A</td>
<td>R</td>
<td>A</td>
</tr>
<tr>
<td>$M_2$</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>$M_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
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</table>
Diagonalization in the proof II

- $D$ outputs opposite of diagonal entries

<table>
<thead>
<tr>
<th></th>
<th>$\langle M_1 \rangle$</th>
<th>$\langle M_2 \rangle$</th>
<th>$\ldots$</th>
<th>$\langle D \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>R</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_2$</td>
<td></td>
<td>R</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td></td>
<td></td>
<td>$\ldots$</td>
<td>$? \quad ?$</td>
</tr>
</tbody>
</table>
co-Turing-recognizable Language I

- Definition: a language is co-Turing-recognizable if its complement is Turing-recognizable
- Theorem 4.22
  Decidable $\iff$ Turing-recognizable and co-Turing-recognizable
- Why not

  Turing-recognizable
  $\Rightarrow$ complement Turing-recognizable

- Note that “recognizable” means any
$w \in$ language

is accepted by the machine in a finite number of steps

That is, no infinite loop

Example:

$A_{TM}$ Turing-recognizable but not decidable

$w \in \overline{A_{TM}}$

$\Rightarrow$ reject or loop

Thus $\overline{A_{TM}}$ is not Turing-recognizable
What if we swap $q_{\text{accept}}, q_{\text{reject}}$?

If $a \notin A$ and loop occurs then $a \in \overline{A}$, but TM still loops.

We cannot reach the new $q_{\text{accept}}$ state.

Proof of Theorem 4.22

“$\Rightarrow$”

Decidable $\Rightarrow$ Turing-recognizable

Complement $\Rightarrow$ decidable $\Rightarrow$ Turing-recognizable
“⇐” Now $A, \bar{A}$ are Turing-recognizable by two machines $M_1, M_2$

Construct a new machine $M$: for any input $w$

1. Run $M_1, M_2$ in parallel
2. $M_1$ accept $\Rightarrow$ accept, $M_2$ accept $\Rightarrow$ reject

Never infinity loop

$M$ accepts all strings in $A$, reject all not in $A$

Thus $A$ is decidable with a decider $M$