There quite a few undecidable problems.

For example, program verification are in general not solvable.

We will discuss an undecidable example called the “halting problem”.

Undecidable problems I
\[ A_{TM} = \{ \langle M, w \rangle \mid M : \text{a TM that accepts } w \} \]

- We will prove that \( A_{TM} \) is undecidable
- However, \( A_{TM} \) is Turing recognizable
- We can simply simulate \( \langle M, w \rangle \)
- To be decidable we hope to avoid an infinite loop
  - if at one point, know it cannot halt
    \( \Rightarrow \) reject
- Thus this problem is called the halting problem
We need a technique called “diagonalization method” for the proof.

It was developed by Cantor in 1873 to check if two infinite sets are equal.

Example: consider

set of even integers

versus

set of \(\{0, 1\}^*\)

Both are infinite sets. Which one is larger?

Definition: two sets are equal if elements can be paired.
Definition 4.12

- \( f \) is a one-to-one function if:

\[
f(a) \neq f(b) \text{ if } a \neq b
\]

- Left: a one-to-one function; right: not
Definition 4.12 II

- $f : A \rightarrow B$ onto if

$$\forall b \in B, \exists a \text{ such that } f(a) = b$$

- Example:

$f(a) = a^2$, where $A = (-\infty, \infty)$ and $B = (-\infty, \infty)$

This is not an onto function because for $b = -1$, there is no $a$ such that $f(a) = b$
However, if we change it to

\[ f(a) = a^2, \text{ where } A = (-\infty, \infty) \text{ and } B = [0, \infty) \]

it becomes an onto function

Definition: a function is called a correspondence if it is one-to-one and onto

Example:

\[ f(a) = a^3, \text{ where } A = (-\infty, \infty) \text{ and } B = (-\infty, \infty) \]
Example 4.13  

- $N = \{1, 2, \ldots\}$
- $E = \{2, 4, \ldots\}$
- The two sets can be paired

\[
\begin{array}{c|c}
 n & f(n) = 2n \\
\hline
 1 & 2 \\
 2 & 4 \\
 \vdots & \vdots \\
\end{array}
\]

- We consider $N$ and $E$ have the same size
- Definition: a set is countable if it is finite or same size as $N$
Rational Numbers Countable

\[ Q = \left\{ \frac{m}{n} \mid m, n \in \mathbb{N} \right\} \text{ countable} \]
Note that we skip counting elements with common factors (e.g., 2/2)
We will use the diagonalization method.

The proof is by contradiction.

Assume \( R \) is countable. Then there is a table as follows:

\[
\begin{array}{|c|c|}
\hline
n & f(n) \\
\hline
1 & 3.14159 \ldots \\
2 & 55.55555 \ldots \\
3 & 0.12345 \ldots \\
4 & 0.50000 \ldots \\
\vdots & \vdots \\
\hline
\end{array}
\]
Consider

\[ x = 0.4641 \ldots \]
\[ 4 \neq 1, 6 \neq 5 \]

We have

\[ x \neq f(n), \forall n \]

But \( x \in \mathbb{R} \), so a contradiction

To avoid the problem

\[ 1 = 0.9999 \ldots \]

for every digit of \( x \) we should not choose 0 or 9