There quite a few undecidable problems

For example, program verification is in general not solvable

We will discuss an undecidable example called the “halting problem”
$A_{TM} = \{\langle M, w \rangle \mid M \text{ : a TM that accepts } w \}$

- We will prove that $A_{TM}$ is undecidable
- However, $A_{TM}$ is Turing recognizable
- We can simply simulate $\langle M, w \rangle$
- To be decidable we hope to avoid an infinite loop if at one point, know it cannot halt
  $\Rightarrow$ reject
- Thus this problem is called the halting problem
Diagonalization method I

- We need a technique called “diagonalization method” for the proof.
- It was developed by Cantor in 1873 to check if two infinite sets are equal.
- Example: consider set of even integers versus set of $\{0, 1\}^*$.
- Both are infinite sets. Which one is larger?
- Definition: two sets are equal if elements can be paired.
Definition 4.12

- $f$ is a one-to-one function if:

$$f(a) \neq f(b) \text{ if } a \neq b$$

- **Left:** a one-to-one function; **right:** not
Definition 4.12 II

- $f : A \rightarrow B$ onto if

$$\forall b \in B, \exists a \text{ such that } f(a) = b$$

- Example:

$$f(a) = a^2, \text{ where } A = (-\infty, \infty) \text{ and } B = (-\infty, \infty)$$

This is not an onto function because for $b = -1$, there is no $a$ such that $f(a) = b$
However, if we change it to

\[ f(a) = a^2, \text{ where } A = (-\infty, \infty) \text{ and } B = [0, \infty) \]

it becomes an onto function

Definition: a function is called a correspondence if it is one-to-one and onto

Example:

\[ f(a) = a^3, \text{ where } A = (-\infty, \infty) \text{ and } B = (-\infty, \infty) \]
Thus a correspondence is a way of pairing elements of a set with elements of another.
Example 4.13 I

- $N = \{1, 2, \ldots\}$
- $E = \{2, 4, \ldots\}$

The two sets can be paired

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n) = 2n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

We consider $N$ and $E$ have the same size

Definition: a set is countable if it is finite or same size as $N$
Rational Numbers Countable

\[ Q = \left\{ \frac{m}{n} \mid m, n \in \mathbb{N} \right\} \text{ countable} \]
Note that we skip counting elements with common factors (e.g., 2/2)
We will use the diagonalization method

The proof is by contradiction

Assume $R$ is countable. Then there is a table as follows

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.14159 ...</td>
</tr>
<tr>
<td>2</td>
<td>55.55555 ...</td>
</tr>
<tr>
<td>3</td>
<td>0.12345 ...</td>
</tr>
<tr>
<td>4</td>
<td>0.50000 ...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Consider

\[ x = 0.4641\ldots \]

\[ 4 \neq 1, 6 \neq 5 \]

We have

\[ x \neq f(n), \forall n \]

But \( x \in R \), so a contradiction

To avoid the problem

\[ 1 = 0.9999\ldots \]

for every digit of \( x \) we should not choose 0 or 9