A language if Turing-recognizable if it is recognized by a TM

For a Turing machine, there are three possible outcomes
- accept, reject, loop

If an input fails: reject or loop
This is difficult to decide

We prefer a TM that never loops
Deciders: only accept or reject
A language is Turing-decidable if some TM decides it.

In Chapter 4 we will discuss decidable languages.
Example 3.9 I

- Consider the following language

\[ \{ w \# w \mid w \in \{0, 1\}^* \} \]

- Fig 3.10
Example 3.9 II

Graph with transitions:
- $0, 1 \rightarrow R$
- $0 \rightarrow x, R$
- $1 \rightarrow x, R$
- $\# \rightarrow R$
- $x \rightarrow R$
- $\# \rightarrow R$
- $0 \rightarrow x, L$
- $1 \rightarrow x, L$
- $x \rightarrow R$
- $0, 1 \rightarrow L$
- $\# \rightarrow L$
- $0, 1, x \rightarrow L$
Example 3.9

- Links to $q_r$ not shown
- Simulate 01#$01$

\[
\begin{align*}
q_101#01 & \quad xq_21#01 & \quad x1q_2#01 & \quad x1#q_401 \\
x1q_6#x1 & \quad xq_71#x1 & \quad q_7x1#x1 & \quad xq_11#x1 \\
xxq_3#x1 & \quad xx#q_5x1 & \quad xx#xq_51 & \quad xx#q_6xx \\
xxq_6#xx & \quad xq_7x#xx & \quad xxq_1#xx & \quad xx#q_8xx \\
xx#xxq_8\sqcup & \quad xx#xx \sqcup q_a
\end{align*}
\]
Example 3.11

\[ C = \{ a^i b^j c^k \mid i \times j = k, i, j, k \geq 1 \} \]

Procedure

1. check if the input is \( a^+ b^+ c^+ \)
2. back to start
3. fix \( a \), for each \( b \), cancel \( c \)
4. store \( b \) back, cancel one \( a \), goto step 3

Too complicated to draw state diagram
But one may wonder if TM can really do the above procedure
Here are more details
Example 3.11 II

- Step 1 can be done by a DFA (as DFA is a special case of TM)
- Step 2 can be done by using a special symbol in the beginning
- Step 3 is similar to the procedure of handling $w\#w$

Now we see the concept of subroutines
Example 3.12 I

- \( E = \{\#x_1\#x_2\cdots\#x_l \mid x_i \in \{0, 1\}^*, x_i \neq x_j\} \)
- Idea: sequentially compare every pair

\[
\begin{align*}
&x_1x_2, x_1x_3, \ldots, x_1x_l \\
&x_2x_3, \ldots, x_2x_l \\
&x_{l-1}x_l
\end{align*}
\]

- Very rough \( \Rightarrow \) more details

For \( x_i, x_j \) mark \#’s of both strings by \( \dot{\#} \)

\( \dot{\#}x_1\#x_2\dot{\#}x_3 \): \( x_1 \) and \( x_3 \) being compared
Example 3.12 II

- Compare $x_i$ and $x_j$:
  - Can use a TM similar to that for $w \neq w$
  - We can copy $x_1, x_2$ to the end and do the comparison there
We discuss some variants that have the same power.

The robustness of a type of machines means that its reasonable variants have the same power.

Not a strict definition though.

Example:

\[ \delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\} \]

\(S\): stay at the same position.
It’s equivalent to TM because $S$ can be implemented by $L$ & $R$ moves:

$q_1, a \rightarrow q_2, b, S$

can be replaced by several rules

$q_1, a \rightarrow q_3, b, R \rightarrow q_2, ?, L, \forall ? \in \Gamma$
Multi-tape TM I

- several tapes
- input: put into tape 1
  others: blank
- transition is applied on all tapes simultaneously

\[ \delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, S, R\}^k \]
\[ \delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, L, R, \ldots, L), \]

where \( k \) is the number of tapes
- Looks more powerful but equivalent
Example I

- Job: given $w = 0^{2n}$, $n \geq 0 \Rightarrow$ generate $ww$ in the end
  
  Note that we also need to check if $|w|$ is even

- State diagram
Example II

- $q_0$ to $q_1$: $0 \rightarrow R$, $\square \rightarrow 0, R$
- $q_1$ to $q_2$: $0 \rightarrow R$, $\square \rightarrow L$
- $q_2$ to $q_3$: $0 \rightarrow R$, $\square \rightarrow L$
- $q_3$ to $q_4$: $\square \rightarrow 0, R$, $0 \rightarrow R$
- $q_4$ to $q_a$: $\square \rightarrow R$, $0 \rightarrow R$
- $q_a$ to $q_0$: $\square \rightarrow 0, R$
Example III

let $\sqcap$ be used to indicate the beginning of the second tape

- loop at $q_1$:
  - copy $w$ to the second tape

- $q_2, q_3$:
  1. move to the beginning of the second tape
  2. check if length is $2n$

- If length $2n$, we should be at $q_3$ instead of $q_2$ when reaching the beginning of the second tape
Example IV

- Example: input 0000

\[
\begin{array}{cccc}
q_0 & 0 & 0 & 0 \\
\text{☐} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & q_2 \\
\hline
\text{☐} & 0 & 0 & q_3 \\
\hline
0 & 0 & 0 & q_4 \\
\hline
0 & q_4 & 0 & 0 \\
\end{array}
\begin{array}{cccc}
q_1 & 0 & 0 & 0 \\
\text{☐} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & q_3 \\
\hline
\text{☐} & 0 & 0 & q_2 \\
\hline
0 & 0 & 0 & q_4 \\
\hline
0 & q_4 & 0 & 0 \\
\end{array}
\begin{array}{cccc}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\end{array}
\begin{array}{cccc}
0 & 0 & 0 & q_1 \\
\text{☐} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & q_2 \\
\hline
\text{☐} & 0 & 0 & q_2 \\
\hline
0 & 0 & 0 & q_4 \\
\hline
0 & q_4 & 0 & 0 \\
\end{array}
\]

accepted

- Example: input 000

\[
\begin{array}{cccc}
q_0 & 0 & 0 & 0 \\
\text{☐} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & q_2 \\
\hline
\text{☐} & 0 & 0 & q_3 \\
\hline
0 & 0 & 0 & q_4 \\
\hline
0 & q_4 & 0 & 0 \\
\end{array}
\begin{array}{cccc}
q_1 & 0 & 0 & 0 \\
\text{☐} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & q_3 \\
\hline
\text{☐} & 0 & 0 & q_2 \\
\hline
0 & 0 & 0 & q_4 \\
\hline
0 & q_4 & 0 & 0 \\
\end{array}
\begin{array}{cccc}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\end{array}
\begin{array}{cccc}
0 & 0 & 0 & q_1 \\
\text{☐} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & q_2 \\
\hline
\text{☐} & 0 & 0 & q_2 \\
\hline
0 & 0 & 0 & q_4 \\
\hline
0 & q_4 & 0 & 0 \\
\end{array}
\]
Example V

rejected
Multi-tape TM $\equiv$ single TM

- Single TM $\subset$ Multiple TM
- But how about the other direction?
- Show single-tape TM can simulate multi-tape TM
- Fig 3.14

```
CPU

0 1 1 0 ...

a a c a ...

0 0 1 1 ...
```
Multi-tape TM $\equiv$ single TM II

- #: a symbol to separate tapes
- $\dot{0}$ is used to store the head position of a tape
- $\Gamma$ becomes different:
  - $\Gamma$ of original multi-tape TM:
    \[ \{0, 1, a, b, \ldots\} \]
Multi-tape TM ≡ single TM III

Γ of new single-tape TM:

\{0, \dot{0}, 1, \dot{1}, a, \dot{a}, b, \dot{b}, \ldots\}

- One multi-tape transition is split to several transitions
  We sequentially conduct them
- What if the transition is “move to right (R)” but we see #?
  \(\Rightarrow\) insert a \(\sqcup\) and shift things after
- How to do the shift? An illustration:
Multi-tape TM ≡ single TM IV

\[ \Gamma \text{ is finite. Use states to remember the current contents} \]