Turing Machines: I

- Part II: computability
  We would like to study problems that can and cannot be solved by computers
- We need more powerful model
  Finite automata: small memory (states)
  PDA: unlimited memory (stack) by push/pop
- Turing machine: unlimited and unrestricted memory
- This is about everything a real computer can do
- Thus problems not solved by Turing machine
  ⇒ beyond the limit of computation
A TM has a tape as the memory

Differences from finite automata
- write/read tape
- head moves left/right
- infinite space in the tape
- rejecting/accepting take immediate effect
- machine goes on forever, otherwise
Example

\[ B = \{ w\#w \mid w \in \{0, 1\}^* \} \]

This language is known to be not a CFL (example 2.22; details not discussed)

Running a sample input. Figure 3.2

\[ \square: \text{blank symbol} \]

We assume infinite \(\square\)'s after the input sequence

Strategy: zig-zag to the corresponding places on the two sides of the \(\#\) and determine whether they match.
Turing Machines: IV

Algorithm:
1. scan to check #
2. check $w$ and $w$
Formal definition of TM I

- It’s complicated and seldom used
- \( \delta: \)
  \[ Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\} \]
- Example:
  \[ \delta(q, a) = (r, b, L) \]
  
  - \( q \): current state
  - \( a \): pointed in tape
  - \( r \): next state
  - \( b \): replace \( a \) with \( b \)
  - \((Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})\)
Formal definition of TM II

\( Q \): states
\( \Sigma \): input alphabet (blank: \( \square \notin \Sigma \))
\( \Gamma \): tape alphabet, \( \square \in \Gamma, \Sigma \subset \Gamma \)
\( \delta \):
\[
Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}
\]
\( q_0 \in Q \), start
\( q_{\text{accept}} \in Q \)
\( q_{\text{reject}} \in Q \), \( q_{\text{reject}} \neq q_{\text{accept}} \)
Single \( q_{\text{accept}}, q_{\text{reject}} \)
The input $w_1 \cdots w_n$ is put in positions 1\ldots, $n$ of the tape in the beginning.

Assume $sqcup$ in all the rest of the tape.
Example 3.7 I

Consider the following language

\[ \{0^{2^n} \mid n \geq 0\} \]

Strings in this language are

0, 00, 0000, 00000000, \ldots

Idea: crossing off every other 0 and the remaining string should still have even length
Example 3.7 II

Example:

0000
00
0

Procedure

1. left → right, mark every other 0
2. if in step 1, only one 0 left, then accept
3. if in step 1, odd ≠ 0 left, then reject
4. move head to the beginning
go back to stage 1

Formally definition

\[ Q = \{ q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}} \} \]

\[ \Sigma = \{ 0 \} \]

\[ \Gamma = \{ 0, x, \square \} \]

The diagram
Example 3.7 V

- $0 \rightarrow R \equiv 0 \rightarrow 0, R$
- Consider the input 0000

$q_10000 \quad \square q_2000 \quad \square xq_300 \quad \square x0q_40 \quad \square x0xq_3$
$
\square x0q_5x \quad \square xq_50x \quad \square q_5x0x \quad q_5 \sqcap x0x \quad \square q_2x0x$
$
\square xq_20x \quad \square xxq_3x \quad \square xxxq_3 \sqcap \quad \square xxq_5x \quad \square xq_5xx$
$
\square q_5xxx \quad q_5 \sqcap xxx \quad \square q_2xxx \quad \square xq_2xx \quad \square xxq_2x$
$
\square xxxq_2 \quad \square xxx \sqcap q_a$

- The $\delta$ function:
No need to have rows for $q_{accept}$, $q_{reject}$

$\Rightarrow$ accepting/rejecting takes immediate effect

Now a deterministic TM

We can have nondeterministic TM later

They are equivalent

Main idea of $\delta$:

$q_1$ : mark the start by $\Box$

- first element must be 0, otherwise, reject
Example 3.7 VII

- Using $\sqcup$, so the start is known
- $q_2 \rightarrow q_3$: handle initial 00
- $q_3 \rightarrow q_4$: sequentially 00 $\rightarrow$ 0x
  - key of this algorithm
  - If not pairs (e.g. 0x0x0x), fails
  - The place where # is even checked
- $q_3 \rightarrow q_5 \rightarrow q_2$ back to beginning
- First 0 (or $\sqcup$) is considered the single final 0

\[ q_2 \rightarrow \cdots \rightarrow q_2 \rightarrow \cdots \rightarrow q_{\text{accept}} \]

check if single 0
The current configuration means current state, tape contents, head location

**uqv**: 
- **q**: current state
- **uv**: current tape content
- **u**: left, **v**: right

**head**: first of **v**
Example of configuration

- $a, b, c \in \Gamma$, $u, v, \in \Gamma^*$ (i.e., strings from $\Gamma$)
- $q_i, q_j$: states
- if $\delta(q_i, b) = (q_j, c, L)$

\[ uaq_ibv \text{ yields } uq_jacv \]

- if $\delta(q_i, b) = (q_j, c, R)$

\[ uaq_ibv \text{ yields } uacq_jv \]
More about Configurations I

- start configuration: $q_0w$
- accepting configuration: $q_{accept}$
- rejecting configuration: $q_{reject}$
- A TM accepts $w$ if configurations $c_1 \cdots c_k$
  1. $c_1$: start configuration
  2. $c_i$ yields $c_{i+1}$
  3. $c_k$ accepting configuration
- Language: $L(M)$: strings accepted by $M$