Informally, an algorithm is a collection of instructions.

Formal definition was not done until 20th century.
In 1900, Hilbert in an address at the International Congress of Mathematicians identified 23 mathematical problems for the coming century.

The 10th asks for an algorithm to test if a polynomial has integer root or not

\[ 6x^3yz^2 + 3xy^2 - x^3 - 10 = 0 \]
\[ x = 5, \ y = 3, \ z = 0 \]

In Hilbert’s description, the word “algorithm” was not used
Hilbert’s problems II

- Roughly he said a process of a finite number of operations.
- However, Hilbert explicitly asked the algorithm be “devised”.
- Thus we need a definition of algorithms.
- In the end this problem is algorithmically unsolvable.
Proposed in 1936
Intuitive algorithms $\equiv$ TM algorithms
Note that this is a definition but not a theorem
Hilbert’s 10th problem I

- Using our terms

\[ D = \{ P \mid P: \text{polynomial with integer roots}\} \]

\(D\): decidable or not?

- A simpler problem of a single variable

\[ D_1 = \{ P \mid P: \text{polynomial of } x \text{ with integer roots}\} \]

- Example:

\[ 4x^3 - 2x^2 + x - 7 \]
Hilbert’s 10th problem II

- We can use a TM to evaluate $x$ at $0, 1, -1, 2, -2, \ldots$
  
  If 0, accept

- If $P$ has no integer root $\Rightarrow$ this evaluation runs forever

- Thus we have a recognizer, but not a decider
It can be proved that roots of a 1-variable polynomial is within the range

$$\pm k \frac{c_{\text{max}}}{c_1}$$

$k$: \# terms, $c_{\text{max}}$: $\max(\text{abs(coefficients)})$
$c_1$: coefficient of the highest order
For the example

\[ 4x^3 - 2x^2 + x - 7 \]

we have

\[ \pm 4 \times \frac{7}{4} = \pm 7 \]

The proof is easy (an exercise in the book)

Unfortunately, the case of multiple variables is very hard

Only until 1970: it’s proved that bounds for multi-variable polynomials are not possible

Thus this problem is undecidable
Description of Turing Machines I

Three levels

1. High-level: no mention how to manage tape and head
   Like how we describe algorithms

2. Implementation-level: English to describe how head moves
For example, our description of the language \( \{ w \# w \mid w \in \{0,1\}^* \} \) is:

- \[ \begin{array}{cccccccc}
0 & 1 & 1 & 0 & 0 & 0 & \# & 0 & 1 & 1 & 0 & 0 & 0 & \square \\
\downarrow & & & & & & & \downarrow \\
\times & 1 & 1 & 0 & 0 & 0 & \# & 0 & 1 & 1 & 0 & 0 & 0 & \square \\
\downarrow & & & & & & & \downarrow \\
\times & 1 & 1 & 0 & 0 & 0 & \# & \times & 1 & 1 & 0 & 0 & 0 & \square \\
\end{array} \]

3 formal-level: all detailed transitions

- We will mainly use high-level descriptions later
Example 3.23 I

\[ A = \{ \langle G \rangle \mid G : a \text{ connected undirected graph} \} \]

- A high-level TM
  1. Mark a node in \( G \)
  2. Repeat until no new nodes marked
     - For every node \( G \), mark it if \( \exists \) an edge to a marked node
  3. If all nodes marked: accept, otherwise: reject
- Real implementation
Example 3.23 II

Figure 3.24

\[ \langle G \rangle = (1, 2, 3, 4)((1, 2), (2, 3), (3, 1), (1, 4)) \]

is the input string

- Details
The first step is to check if the input is in the correct format.

In the first step we begin by seeing if the first part of the input $\langle G \rangle$ includes distinct numbers (as node IDs should be different).

This is similar to an example before:

$$\{\#x_1\#x_2 \cdots \#x_l \mid x_i \in \{0,1\}^*, x_i \neq x_j \}$$

Then we can talk about how the head is moved.

Thus we have implementation-level descriptions.