Lemma 2.27
Language recognized by PDA $\Rightarrow$ context free

Idea:

any states $p, q$ of a PDA $P$
$\Rightarrow$ we have a variable $A_{pq}$

and

$A_{pq}$ generates $x \iff$
$P$ from $p$ with empty stack to $q$ with empty stack
Need to modify P so that:

1. Single accept: $q_{\text{accept}}$
   Then $A_{q_{\text{start}}, q_{\text{accept}}}$ is the start variable to generate any string $x$ of this language.

2. Stack should be empty before accepting
   In the beginning stack is empty and we need this property to have (1).

3. Each transition push or pop, but not both

We will explain how to make the PDA satisfy these conditions.
Now we focus on the more important part: construction of the rules

For (1) we don’t really mean “empty stack.” We actually mean “stack with the same contents.”

For the following figure, rules

\[ A_{pq} \rightarrow A_{pr}A_{rq}, \forall p, q, r \in Q \]

should be generated

- x-axis: input string
- y-axis: stack height
Reason: If we can go from $p$ to $r$ without changing stack and from $r$ to $q$ without changing stack
then we can do
from $p$ to $q$ without changing stack

In the following figure we have

$p, q, r, s \in Q$, $t \in \Gamma$, $a, b \in \Sigma_{\epsilon}$

If

$$(r, t) \in \delta(p, a, \epsilon), (q, \epsilon) \in \delta(s, b, t)$$

then we should have

$$A_{pq} \rightarrow aA_{rs}b$$
Finally we need

\[ A_{pp} \rightarrow \epsilon, \forall p \in Q \]

Let’s discuss an example first
Examples I

- \( \{0^n1^n \mid n \geq 1\} \)

- This is modified from an earlier example. Now \( q_1 \) is not an accept state

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\begin{align*}
q_1 \xrightarrow{\epsilon, \epsilon} \$ & & q_2 \xrightarrow{0, \epsilon} 0 \\
q_4 \xrightarrow{\epsilon, \$} \epsilon & & q_3 \xrightarrow{1, 0} \epsilon
\end{align*}
```
Examples II

- Three conditions satisfied
  - Each transition push or pop only
- \( t = \$ \)

\[
\begin{array}{cccccccc}
  p & r & s & q & t & a & b \\
  1 & 2 & 3 & 4 & \$ & \epsilon & \epsilon \\
\end{array}
\]

rule:

\[
A_{14} \rightarrow \epsilon A_{23} \epsilon
\]

- \( t = 0 \)

\[
\begin{array}{cccccccc}
  p & r & s & q & t & a & b \\
  2 & 2 & 2 & 3 & 0 & 0 & 1 \\
  2 & 2 & 3 & 3 & 0 & 0 & 1 \\
\end{array}
\]
Examples III

rules:

\[ A_{23} \rightarrow 0A_{22}1 \]
\[ A_{23} \rightarrow 0A_{23}1 \]

- Other rules: 64 rules

\[ A_{11} \rightarrow A_{11}A_{11} \]
\[ A_{11} \rightarrow A_{12}A_{21} \]
\[ A_{11} \rightarrow A_{13}A_{31} \]
\[ A_{11} \rightarrow A_{14}A_{41} \]
\[ \vdots \]
Examples IV

and

\[
A_{11} \rightarrow \epsilon \\
A_{22} \rightarrow \epsilon \\
A_{33} \rightarrow \epsilon \\
A_{44} \rightarrow \epsilon
\]