We are interested in the limit of finite automata

- Some languages not recognized

\[ \{0^n1^n \mid n \geq 0\} \]

- We might remember #0 first
  - But # of possible \( n \)'s is \( \infty \)
- Thus we cannot recognize it by finite automata
- However, this is not a formal proof
- It may be difficult to quickly tell if a language is regular or not
Consider two languages

\[ C = \{ w \mid \#0 = \#1 \} \]
\[ D = \{ w \mid \#01 = \#10 \} \]

It seems that both are not regular

Indeed, \( C \) is not regular but \( D \) is

This is an exercise in the book, so we don’t give details

To formally prove a language is not regular, we will introduce the pumping lemma
Pumping lemma I

- Strategy: by contradiction
- We prove
  \[ \text{regular} \Rightarrow \text{some properties} \]
- If “some properties” cannot hold, then the language is not regular
Theorem 1.70 I

If $A$ regular $\Rightarrow \exists p$ (pumping length) such that $\forall s \in A, |s| \geq p$, $\exists x, y, z$ such that $s = xyz$ and

1. $\forall i \geq 0, xy^i z \in A$
2. $|y| > 0$
3. $|xy| \leq p$

Note that for $y^i$, we have $y^0 = \epsilon$
Proof of pumping lemma I

Because $A$ is regular, $\exists$ a DFA to recognize $A$
Let $p = \#$ states of this DFA
If no string $s$ such that $|s| \geq p$, then the theorem statement is satisfied
Now consider any $s$ with $|s| \geq p$

$$s = s_1 \cdots s_n$$
Proof of pumping lemma II

- To process this string, assume the state sequence is

  \[ q_1 \cdots q_{n+1} \]

  Because \(|s| \geq p\), we have

  \[ n + 1 > p \]

- In \(1 \ldots p + 1\) two states must be the same
  (pigeonhole principle)

Fig 1.72
Proof of pumping lemma III
Proof of pumping lemma IV

Assume

\[ q_j \text{ and } q_l \text{ with } j \leq p + 1, \ l \leq p + 1 \]

are two same states. Then

\[ x = s_1 \cdots s_{j-1}, \ y = s_j, \cdots s_{l-1}, \ z = s_l \cdots s_n \]

Because \( j \neq l \),

\[ |y| > 0 \]

From \( l \leq p + 1 \), we have

\[ |xy| \leq p \]

Thus all conditions are satisfied
Let’s apply pumping lemma to prove that

\[ B = \{0^n1^n \mid n \geq 0\} \]

is not regular

Assume \( B \) is regular. From the lemma there is \( p \) such that

\[ s = 0^p1^p = xyz \]

By the lemma, \(|y| > 0 \) and \( xy^iz \in B, \forall i \geq 0 \)
Example 1.73 II

1. If

\[ y = 0 \cdots 0 \]

then

\[ xy = 0 \cdots 0 \quad \text{and} \quad z = 0 \cdots 01 \cdots 1 \]

Thus

\[ xyyz : \#0 > \#1 \]

2. Then \( xy^2 z \notin B \), a contradiction
Example 1.73 III

3. If

\[ y = 1 \cdots 1, \]

similarly

\[ xy^2z \notin B \text{ as } \#0 < \#1 \]

4. If

\[ y = 0 \cdots 01 \cdots 1 \]

then

\[ xyyz \notin B \text{ as it is not in the form of } 0^?1^? \]
We see that the condition

$$|xy| \leq p$$

is not used.