Closure under regular operations

Recall we define three operations: $\cup$, $\circ$, $\ast$

We will see that by using NFA, the proof is easier
Given two regular languages $A_1, A_2$ under the same $\sum$.

Is $A_1 \cup A_2$ regular?

To prove that a language is regular, by definition, it should be accepted by one DFA (or an NFA).

We will construct an NFA for $A_1 \cup A_2$.

Assume $A_1$ and $A_2$ are recognized by two NFAs $N_1$ and $N_2$, respectively.
Formal definition
Two NFAs:

\[ N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \]
\[ N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \]

Note for NFA, \( \epsilon \notin \Sigma \)
New NFA

\[ Q = Q_1 \cup Q_2 \cup \{ q_0 \} \]

\[ q = q_0 \]

\[ F = F_1 \cup F_2 \]

\[ \delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \\
\delta_2(q, a) & q \in Q_2 \\
\{ q_1, q_2 \} & q = q_0 \text{ and } a = \epsilon \\
\emptyset & q = q_0 \text{ and } a \neq \epsilon 
\end{cases} \]
The last case of $\delta$ is easily neglected
Closed Under Concatenation I

Given two NFAs

$N_1$ $N_2$

- Idea: from any accept state of $N_1$, add an $\epsilon$ link to $q_2$ (start state of $N_2$)
- The new machine:
Formal definition. Given two automata

\[(Q_1, \Sigma, \delta_1, q_1, F_1)\]

\[(Q_2, \Sigma, \delta_2, q_2, F_2)\]
Closed Under Concatenation III

- New machine

\[ Q = Q_1 \cup Q_2 \]

\[ q_0 = q_1 \]

\[ F = F_2 \]

\( \delta \) function:

\[ \delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \setminus F_1 \\
\delta_2(q, a) & q \in Q_2 \\
\delta_1(q, \epsilon) \cup \{q_2\} & q \in F_1, a = \epsilon \\
\delta_1(q, a) & q \in F_1, a \neq \epsilon 
\end{cases} \]
Closed under star I

- Given the following machine

- Recall the star operation is defined as follows

\[ A^* = \{ x_1 \cdots x_k \mid k \geq 0, x_i \in A \} \]

- How about the following diagram
The problem is that $\epsilon$ may not be accepted.

How about making the start state an accepting one?
Closed under star III

- This may make the machine to accept strings not in $A$
- A correct setting

Formal definition

Given the machine $(Q_1, \Sigma, \delta_1, q_1, F_1)$
Closed under star IV

- New machine:

\[ Q = Q_1 \cup \{ q_0 \} \]
\[ q_0 : \text{new start state} \]
\[ F = F_1 \cup \{ q_0 \} \]

\[ \delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \setminus F_1 \\
\delta_1(q, a) \cup \{ q_1 \} & q \in F_1, a = \epsilon \\
\delta_1(q, a) & q \in F_1, a \neq \epsilon \\
\{ q_1 \} & q = q_0, a = \epsilon \\
\emptyset & q = q_0, a \neq \epsilon 
\end{cases} \]