Recall we define three operations: 

\( \cup, \circ, * \)

We will see that by using NFA, the proof is easier.
Given two regular languages $A_1, A_2$ under the same $\Sigma$.

Is $A_1 \cup A_2$ regular?

To prove that a language is regular, by definition, it should be accepted by one DFA (or an NFA).

We will construct an NFA for $A_1 \cup A_2$.

Assume $A_1$ and $A_2$ are recognized by two NFAs $N_1$ and $N_2$, respectively.
Formal definition
Two NFAs:

\[ N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \]
\[ N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \]

Note for NFA, \( \epsilon \notin \Sigma \)
New NFA

\[ Q = Q_1 \cup Q_2 \cup \{q_0\} \]
\[ q = q_0 \]
\[ F = F_1 \cup F_2 \]

\[ \delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \\
\delta_2(q, a) & q \in Q_2 \\
\{q_1, q_2\} & q = q_0 \text{ and } a = \epsilon \\
\emptyset & q = q_0 \text{ and } a \neq \epsilon 
\end{cases} \]
The last case of $\delta$ is easily neglected
Given two NFAs

$N_1$

$N_2$

- Idea: from any accept state of $N_1$, add an $\epsilon$ link to $q_2$ (start state of $N_2$)
- The new machine:
Closed Under Concatenation II

Formal definition. Given two automata

\[(Q_1, \Sigma, \delta_1, q_1, F_1)\]
\[(Q_2, \Sigma, \delta_2, q_2, F_2)\]
Closed Under Concatenation III

- New machine

\[ Q = Q_1 \cup Q_2 \]
\[ q_0 = q_1 \]
\[ F = F_2 \]

\( \delta \) function:

\[ \delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \setminus F_1 \\
\delta_2(q, a) & q \in Q_2 \\
\delta_1(q, \epsilon) \cup \{q_2\} & q \in F_1, a = \epsilon \\
\delta_1(q, a) & q \in F_1, a \neq \epsilon 
\end{cases} \]
Given the following machine

Recall the star operation is defined as follows

\[ A^* = \{ x_1 \cdots x_k \mid k \geq 0, x_i \in A \} \]

How about the following diagram
The problem is that $\epsilon$ may not be accepted.
How about making the start state an accepting one?
Closed under star III

- This may make the machine to accept strings not in $A$
- A correct setting
Closed under star IV

- Formal definition

\[(Q_1, \Sigma, \delta_1, q_1, F_1)\]

\[Q = Q_1 \cup \{q_0\}\]

\[q_0: \text{new start state}\]

\[F = F_1 \cup \{q_0\}\]

\[\delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \setminus F_1 \\
\delta_1(q, a) \cup \{q_1\} & q \in F_1, a = \epsilon \\
\delta_1(q, a) & q \in F_1, a \neq \epsilon \\
\{q_1\} & q = q_0, a = \epsilon \\
\emptyset & q = q_0, a \neq \epsilon 
\end{cases}\]
Closed under star V