Recall we define three operations: \( \cup, \circ, \ast \)

We will see that by using NDA, the proof is easier
Given two regular languages $A_1, A_2$ under the same $\Sigma$
Is $A_1 \cup A_2$ regular?
To prove that a language is regular, by definition, it should be accepted by one DFA (or an NFA)
We will construct an NFA for $A_1 \cup A_2$
Assume $A_1$ and $A_2$ are recognized by two NFAs $N_1$ and $N_2$, respectively
Formal definition
Two NFAs:

\[ N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \]
\[ N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \]

Note for NFA, \( \epsilon \notin \Sigma \)
New NFA

\[ Q = Q_1 \cup Q_2 \cup \{ q_0 \} \]
\[ q = q_0 \]
\[ F = F_1 \cup F_2 \]

\[ \delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \\
\delta_2(q, a) & q \in Q_2 \\
\{ q_1, q_2 \} & q = q_0 \text{ and } a = \varepsilon \\
\emptyset & q = q_0 \text{ and } a \neq \varepsilon 
\end{cases} \]
The last case of $\delta$ is easily neglected
Closed Under Concatenation I

Given two NFAs

\[ N_1 \]

\[ N_2 \]

- Idea: from any accept state of \( N_1 \), add an \( \epsilon \) link to \( q_2 \) (start state of \( N_2 \))
- The new machine:
Formal definition. Given two automata

\[(Q_1, \Sigma, \delta_1, q_1, F_1)\]
\[(Q_2, \Sigma, \delta_2, q_2, F_2)\]
Closed Under Concatenation III

- New machine

\[ Q = Q_1 \cup Q_2 \]
\[ q_0 = q_1 \]
\[ F = F_2 \]

\( \delta \) function:

\[ \delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \setminus F_1 \\
\delta_2(q, a) & q \in Q_2 \\
\delta_1(q, \epsilon) \cup \{q_2\} & q \in F_1, a = \epsilon \\
\delta_1(q, a) & q \in F_1, a \neq \epsilon 
\end{cases} \]
Given the following machine

Recall the star operation is defined as follows

\[ A^* = \{ x_1 \cdots x_k \mid k \geq 0, x_i \in A \} \]

How about the following diagram
The problem is that $\epsilon$ may not be accepted.
How about making the start state an accepting one?
Closed under star \( \text{III} \)

- This may make the machine to accept strings not in \( A \)
- A correct setting
Closed under star IV

- Formal definition

\[(Q_1, \Sigma, \delta_1, q_1, F_1)\]

\[Q = Q_1 \cup \{q_0\}\]

\[q_0: \text{new start state}\]

\[F = F_1 \cup \{q_0\}\]

\[\delta:\]

\[\delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \setminus F_1 \\
\delta_1(q, a) \cup \{q_1\} & q \in F_1, a = \epsilon \\
\delta_1(q, a) & q \in F_1, a \neq \epsilon \\
\{q_1\} & q = q_0, a = \epsilon
\end{cases}\]
Closed under star V