Set
Omitted

Sequence and tuples

- Sequence: Objects in order
  
  \[(7, 21, 57) \neq (57, 7, 21)\]

- Repetition

  set: \(\{7, 21, 57\} = \{7, 7, 21, 57\}\)

  sequences: \(\{7, 21, 57\} \neq \{7, 7, 21, 57\}\)
Tuples: finite sequence
(7, 21, 57): 3-tuple

Cartesian product:

\[ A = \{1, 2\}, \quad B = \{x, y\} \]
\[ A \times B = \{(1, x), (1, y), (2, x), (2, y)\} \]

Function: single output

Relation: scissors-paper-stone

<table>
<thead>
<tr>
<th></th>
<th>beats</th>
<th>scissors</th>
<th>paper</th>
<th>stone</th>
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Equivalence relation

1. Reflexive
   \[ \forall x, xRx \]

2. Symmetric
   \[ xRy \iff yRx \]

3. Transitive
   \[ xRy, yRz \Rightarrow xRz \]

E.g. “=”
Example: \( i \equiv_7 j \) if \( 0 = i - j \mod 7 \)

\[
\begin{align*}
  i - i \mod 7 &= 0 \\
  i - j &= 7a, j - i = -7a \\
  i - j &= 7a, j - k = 7b \\
  \Rightarrow i - k &= 7(a + b)
\end{align*}
\]

Graph
- Undirected
- Directed
Nodes (vertices)

- Edges: connection between nodes

Degree = \# edges at a node

Subgraph: $G$ subgraph of $H$ if

- $G$ a graph
- $\text{node}(G) \subset \text{node}(H)$
- $\text{edge}(G) =$ subset of $\text{edge}(H)$ connecting $\text{node}(G)$

In our example,
is a subgraph, but

is not

- Strings and languages
  - alphabet: \{0, 1\}
  - string: 1001
  - language: set of strings

- Boolean logic
  - true and false
Mathematical notions VII

- 0 (false) and 1 (true)
- $0 \land 0 = 0$, $0 \lor 0 = 0$, $\neg 0 = 1$ (negation operation)
- xor $\oplus$

- $0 \otimes 0 = 0$
- $0 \otimes 1 = 1$
- $1 \otimes 0 = 1$
- $1 \otimes 1 = 0$

- implication
Mathematical notions VIII

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \rightarrow Q$</th>
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Why

$P = 0, Q = 1$, then $P \rightarrow Q = 1$

Consider

rainy $\rightarrow$ wet land

If not rainy, saying rainy implies wet land is ok.
### Mathematical notions IX

**$P \rightarrow Q \equiv \neg P \lor Q$**

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<tr>
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<th>$P \rightarrow Q$</th>
<th>$\neg P$</th>
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Proof I

- Direct proof:
  \[ A \rightarrow B \]

- Proof by contradiction
  \[ \neg B \rightarrow \neg A \]

- Example 1:
  Every graph \( \Rightarrow \) sum of degrees is even
  - An example:
    \[ \text{\# degrees} = 1 + 2 + 1 = 4 \]
Proof II

- Each edge: 2 nodes
  \[ \text{total } \# \text{ degrees } = 2 \times \# \text{ edges} \]

Example 2: \( \sqrt{2} \) is irrational

The implication

Definition of rational number

\[ \Rightarrow \sqrt{2} \text{ is not rational} \]

That is,

If a rational number is ...

\[ \Rightarrow \sqrt{2} \text{ is not rational} \]
The opposite is

If $\sqrt{2}$ is not rational
⇒ The rational number cannot be defined as ...

- If not

$$\sqrt{2} = \frac{m}{n}$$

and $m, n$ have no common factor
Then

\[2n^2 = m^2\]

Looks impossible. But how to write this formally?

First we prove that \(m\) must be even. This is also by contradiction
If \( m \) is not even,
\[
m = 2k + 1.
\]

Then
\[
m^2 = 4(k^2 + k) + 1
\]
is not even and
\[
m^2 = 2n^2
\]
does not hold.
Now $m$ even

$m = 2k$

Then

$n^2 = 2k^2$

By the same argument, $n$ is even

Thus $m, n$ have a common factor 2 and there is a contradiction