Mathematical notions I

- Set
  Omitted
- Sequence and tuples
  - Sequence: Objects in order
    \[(7, 21, 57) \neq (57, 7, 21)\]
- Repetition

set: \(\{7, 21, 57\} = \{7, 7, 21, 57\}\)
sequences: \((7, 21, 57) \neq (7, 7, 21, 57)\)
Mathematical notions II

- **Tuples**: finite sequence
  \((7,21,57)\): 3-tuple

- **Cartesian product**:
  \[ A = \{1, 2\}, \quad B = \{x, y\} \]
  \[ A \times B = \{(1, x), (1, y), (2, x), (2, y)\} \]

- **Function**: single output

- **Relation**: scissors-paper-stone

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<thead>
<tr>
<th></th>
<th>scissors</th>
<th>paper</th>
<th>stone</th>
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<td>beats</td>
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Equivalence relation

1. Reflexive
   \[ \forall x, xRx \]

2. Symmetric
   \[ xRy \iff yRx \]

3. Transitive
   \[ xRy, yRz \Rightarrow xRz \]

E.g. "="
Example: \( i \equiv_7 j \) if \( 0 = i - j \mod 7 \)

\[
i - i \mod 7 = 0
\]

\[
i - j = 7a, j - i = -7a
\]

\[
i - j = 7a, j - k = 7b
\]

\[
\Rightarrow i - k = 7(a + b)
\]

Graph

Undirected

Directed
Nodes (vertices)

- Edges: connection between nodes
- Degree = \# edges at a node
- Subgraph: \( G \) is subgraph of \( H \) if
  - \( G \) is a graph
  - \( \text{node}(G) \subset \text{node}(H) \)
  - \( \text{edge}(G) \) = subset of \( \text{edge}(H) \) connecting \( \text{node}(G) \)

In our example,
Mathematical notions VI

is a subgraph, but

is not

- Strings and languages
  - alphabet: \{0, 1\}
  - string: 1001
  - language: set of strings

- Boolean logic
  - true and false
0 (false) and 1 (true)
$0 \land 0 = 0, 0 \lor 0 = 0, \neg 0 = 1$ (negation operation)
xor $\otimes$

\[
\begin{align*}
0 \otimes 0 &= 0 \\
0 \otimes 1 &= 1 \\
1 \otimes 0 &= 1 \\
1 \otimes 1 &= 0
\end{align*}
\]

implication
Mathematical notions VIII

<table>
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<th>Q</th>
<th>P → Q</th>
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Why

\[ P = 0, \ Q = 1, \ \text{then} \ P \rightarrow Q = 1 \]

Consider

rainy \rightarrow \text{wet land}

If not rainy, saying rainy implies wet land is ok.
\[ P \rightarrow Q \equiv \neg P \lor Q \]

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Proof I

- Direct proof:
  \[ A \rightarrow B \]

- Proof by contradiction
  \[ \neg B \rightarrow \neg A \]

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Example 1:
Every graph $\Rightarrow$ sum of degrees is even
- An example:

```
  1 -- 2 -- 3
```

$\# \text{ degrees} = 1 + 2 + 1 = 4$
- Each edge: 2 nodes

total $\# \text{ degrees} = 2 \times \# \text{ edges}$

Example 2: $\sqrt{2}$ is irrational
Proof III

- The implication

  Definition of rational numbers
  \[ \Rightarrow \sqrt{2} \text{ is not rational} \]

  That is,

  If a rational number is ...
  \[ \Rightarrow \sqrt{2} \text{ is not rational} \]

  The opposite is

  If \( \sqrt{2} \) is rational
  \[ \Rightarrow \text{The rational number cannot be defined as ...} \]
Proof IV

- If $\sqrt{2}$ is rational
  
  $$\sqrt{2} = \frac{m}{n}$$
  
  and $m$, $n$ have no common factor
- Then
  
  $$2n^2 = m^2$$

  Looks impossible. But how to write this formally?
- First we prove that $m$ must be even. This is also proof by contradiction
Proof V

If $m$ is not even,

$$m = 2k + 1.$$  

Then

$$m^2 = 4(k^2 + k) + 1$$

is not even and

$$m^2 = 2n^2$$

does not hold.
Now suppose $m$ is even

$$m = 2k$$

Then

$$n^2 = 2k^2$$

By the same argument, $n$ is even

Thus $m$, $n$ have a common factor 2 and there is a contradiction