Mathematical notions I

- Set
  Omitted
- Sequence and tuples
  - Sequence: Objects in order
    
    $$(7, 21, 57) \neq (57, 7, 21)$$

- Repetition
  
  set : $\{7, 21, 57\} = \{7, 7, 21, 57\}$
  sequences : $\{7, 21, 57\} \neq \{7, 7, 21, 57\}$
Mathematical notions II

- Tuples: finite sequence
  (7,21,57): 3-tuple
- Cartesian product:

  \[ A = \{1, 2\}, \quad B = \{x, y\} \]

  \[ A \times B = \{(1, x), (1, y), (2, x), (2, y)\} \]

- Function: single output
- Relation: scissors-paper-stone

<table>
<thead>
<tr>
<th></th>
<th>beats</th>
<th>scissors</th>
<th>paper</th>
<th>stone</th>
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Equivalence relation

1. reflexive
   \[ \forall x, xRx \]

2. symmetric
   \[ xRy \iff yRx \]

3. transitive
   \[ xRy, yRz \Rightarrow xRz \]

e.g. “=”
Example: \( i \equiv_7 j \) if \( 0 = i - j \mod 7 \)

\[
\begin{align*}
i - i & \mod 7 = 0 \\
i - j & = 7a, j - i = -7a \\
i - j & = 7a, j - k = 7b \\
\Rightarrow i - k & = 7(a + b)
\end{align*}
\]

Graph

Undirected

\[ 
\begin{array}{ccc}
  \circ & \circ & \circ \\
\end{array}
\]

Directed
Mathematical notions V

Nodes (vertices)

- Edges: connection between nodes
- Degree = \# edges at a node

Subgraph: $G$ is subgraph of $H$ if
- $G$ is a graph
- $\text{node}(G) \subset \text{node}(H)$
- $\text{edge}(G) = \text{subset of} \text{edge}(H) \text{ connecting} \text{node}(G)$

In our example,
is a subgraph, but

is not

- Strings and languages
  - alphabet: \{0, 1\}
  - string: 1001
  - language: set of strings

- Boolean logic
  - true and false
Mathematical notions VII

- 0 (false) and 1 (true)
- $0 \land 0 = 0, \ 0 \lor 0 = 0, \ \neg 0 = 1$ (negation operation)
- xor $\otimes$
  
  $\begin{align*}
  0 \otimes 0 & = 0 \\
  0 \otimes 1 & = 1 \\
  1 \otimes 0 & = 1 \\
  1 \otimes 1 & = 0
  \end{align*}$

- implication
Mathematical notions VIII

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \rightarrow Q$</th>
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Why

$P = 0, Q = 1$, then $P \rightarrow Q = 1$

Consider

rainy $\rightarrow$ wet land

If not rainy, saying rainy implies wet land is ok.
\[ P \rightarrow Q \equiv \neg P \lor Q \]

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<th>( \neg P \lor Q )</th>
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Proof I

- Direct proof:
  
  \[ A \rightarrow B \]

- Proof by contradiction
  
  \[ \neg B \rightarrow \neg A \]

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Proof II

- Example 1:
  Every graph $\Rightarrow$ sum of degrees is even
  - An example:
    
    \[
    \begin{array}{c}
    \text{○} - \text{○} - \text{○}
    \end{array}
    \]
    
    \# degrees = 1 + 2 + 1 = 4
  - Each edge: 2 nodes
    
    \[
    \text{total \# degrees} = 2 \times \# \text{edges}
    \]
  - Example 2: $\sqrt{2}$ is irrational
Proof III

- The implication

  Definition of rational numbers

  \( \Rightarrow \sqrt{2} \) is not rational

  That is,

  If a rational number is ...

  \( \Rightarrow \sqrt{2} \) is not rational

  The opposite is

  If \( \sqrt{2} \) is rational

  \( \Rightarrow \) The rational number cannot be defined as ...
If $\sqrt{2}$ is rational

$$\sqrt{2} = \frac{m}{n}$$

and $m, n$ have no common factor.

Then

$$2n^2 = m^2$$

Looks impossible. But how to write this formally?

First we prove that $m$ must be even. This is also proof by contradiction.
Proof V

If $m$ is not even,

$$m = 2k + 1.$$ 

Then

$$m^2 = 4(k^2 + k) + 1$$

is not even and

$$m^2 = 2n^2$$

does not hold.
Now suppose $m$ is even

$$m = 2k$$

Then

$$n^2 = 2k^2$$

By the same argument, $n$ is even

Thus $m, n$ have a common factor 2 and there is a contradiction