SDP Relaxation of Quadratic Optimization with Few Homogeneous Quadratic Constraints

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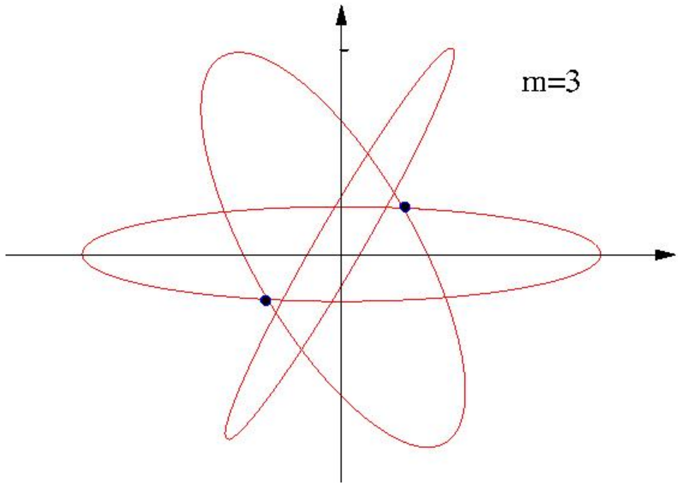
Talk Outline

- Problem description & motivation
- SDP relaxation
- Approximation upper and lower bounds
- Proof idea
- Numerical experience
- A related problem
- Conclusions & open questions

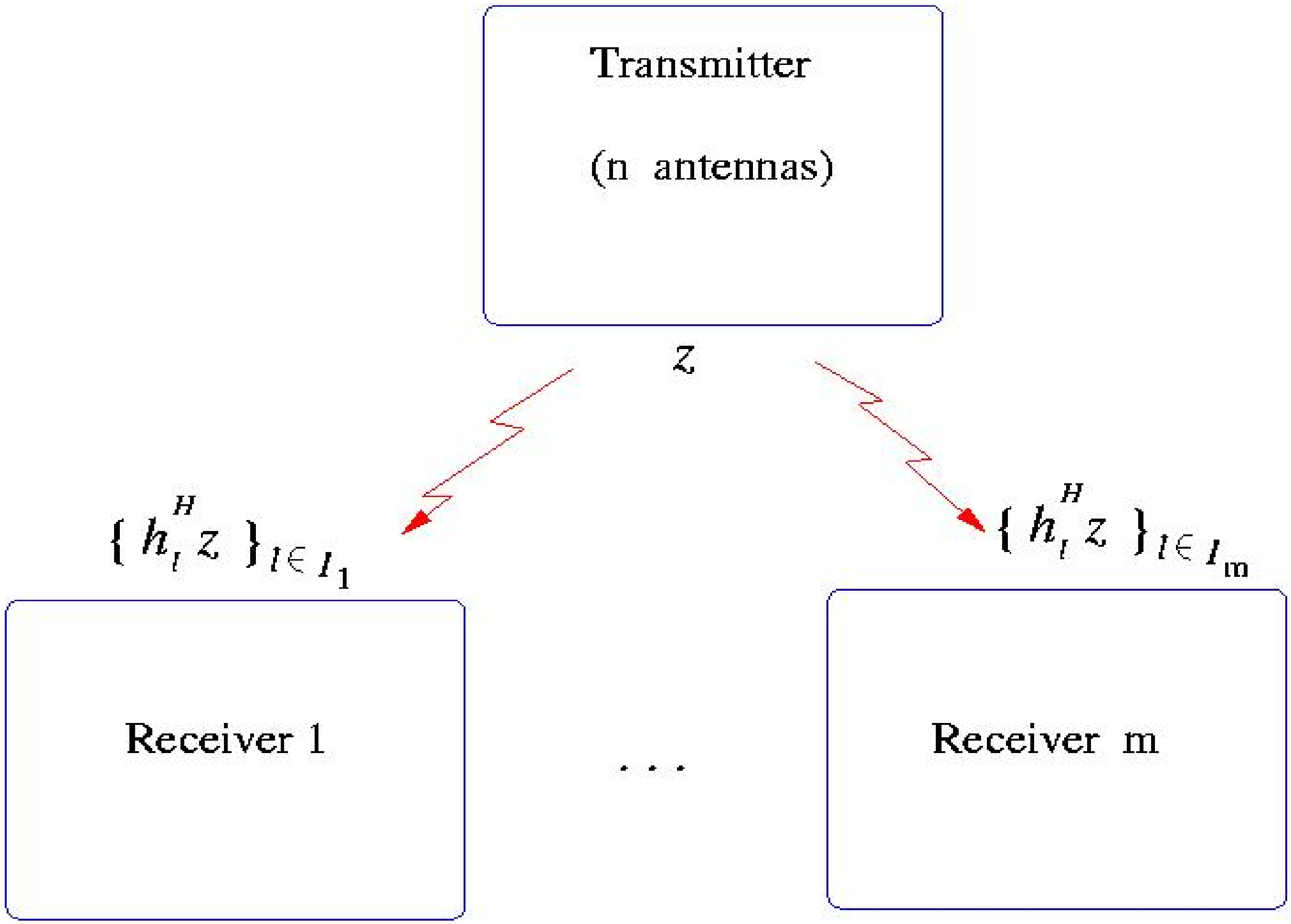
Problem description

$$v_{\rm qp} := \min_{z\in\mathcal{H}} \quad \|z\|^2$$
$$\text{s.t.} \quad \sum_{\ell\in I_i} |h_\ell^H z|^2 \geq 1, \quad i = 1, \ldots, m,$$

- $h_\ell \neq 0 \in \mathcal{H}$ ($\mathcal{H} = \mathbb{C}^n$ or $\mathbb{R}^n$), $\quad I_1 \cup \cdots \cup I_m = \{1, \ldots, M\}$
- $z = x + iy$ ($x, y \in \mathbb{R}^n$), $\quad z^H = x^T - iy^T$



Motivation: Transmit beam forming



SDP Relaxation

- Finding a global minimum of QP is NP-hard (reduction from PARTITION).

- Approximate QP by an "easy" convex optimization problem, a semidefinite program (SDP) relaxation (Lovász '91, Shor '87).

SDP Relaxation

Let $Z = zz^H$ $\quad (\Longleftrightarrow Z \succeq 0, \text{rank}Z \leq 1)$ $\quad H_i = \sum_{\ell \in I_i} h_\ell h_\ell^H$

$$\begin{aligned} v_{\text{qp}} \;=\; \min \quad & \text{Tr}(Z) \\ \text{s.t.} \quad & \text{Tr}(H_i Z) = \sum_{\ell \in I_i} \text{Tr}(h_\ell h_\ell^H Z) \geq 1, \quad i = 1, ..., m, \\ & Z \succeq 0, \quad \text{rank}Z \leq 1. \end{aligned}$$

$$\begin{aligned} v_{\text{sdp}} := \min \quad & \text{Tr}(Z) \\ \text{s.t.} \quad & \text{Tr}(H_i Z) \geq 1, \quad i = 1, ..., m, \\ & Z \succeq 0. \end{aligned}$$

Then

$$0 \;\leq\; v_{\text{sdp}} \;\leq\; v_{\text{qp}} \;\overset{?}{\leq}\; C v_{\text{sdp}} \qquad (C \geq 1)$$

Approximation upper & lower bounds

Theorem 1 (LSTZ '05): $v_{\rm qp} \le C v_{\rm sdp}$ where

$$\frac{1}{2\pi^2} m^2 \le C \le \frac{27}{\pi} m^2 \qquad \text{if} \quad \mathcal{H} = \mathbb{R}^{\rm n}$$

$$\frac{1}{2(3.6\pi)^2} m \le C \le 8m \qquad \text{if} \quad \mathcal{H} = \mathbb{C}^{\rm n}$$

Proof sketch

$\mathcal{H} = \mathbb{R}^n$

Let Z^* be an optimal SDP soln, with rank $r \leq \sqrt{2m}$ (such Z^* exists).

So $Z^* = \sum_{k=1}^r z_k z_k^H$ ($z_k \in \mathcal{H}$)

Let $\zeta := \sum_{k=1}^r z_k \eta_k, \quad \eta_k \overset{\text{i.i.d.}}{\sim} N(0,1)$

Fact:

- $\mathrm{E}(\zeta^H H_i \zeta) = \mathrm{Tr}(H_i Z^*) \geq 1 \; \forall \, i$
- $\mathrm{E}(\|\zeta\|^2) = \mathrm{Tr}(Z^*)$
- $\mathrm{P}(\zeta^H H_i \zeta < \gamma) \leq \sqrt{\gamma} \quad \forall \, \gamma > 0, \; \forall \, i$ $\qquad \left(\mathrm{P}(|\eta_k|^2 < \gamma) \leq \sqrt{\frac{2\gamma}{\pi}} \right)$
- $\mathrm{P}(\|\zeta\|^2 > \mu \mathrm{Tr}(Z^*)) \leq \frac{1}{\mu} \; \forall \, \mu > 0$ (Markov ineq.)

$$\begin{aligned}
& \mathrm{P}\left(\zeta^H H_i \zeta \geq \gamma,\ i=1,...,m \ \&\ \|\zeta\|^2 \leq \mu \mathrm{Tr}(Z^*)\right) \\
\geq \quad & 1 - \sum_{i=1}^{m} \mathrm{P}\left(\zeta^H H_i \zeta < \gamma\right) - \mathrm{P}\left(\|\zeta\|^2 > \mu \mathrm{Tr}(Z^*)\right) \\
\geq \quad & 1 - m\sqrt{\gamma} - \frac{1}{\mu} \\
> \quad & 0 \qquad \text{if } \mu = 3,\ \gamma = \frac{\pi}{9m^2}
\end{aligned}$$

so $\exists\ \zeta \in \Re^n$ such

$$\zeta^H H_i \zeta \geq \frac{\pi}{9m^2},\ i=1,...,m \qquad \|\zeta\|^2 \leq 3\mathrm{Tr}(Z^*) = 3\upsilon_{\mathrm{sdp}}.$$

Then $\hat{z} := \frac{\zeta}{\sqrt{\min_i \zeta^H H_i \zeta}}$ is a feas. soln of QP, $\|\hat{z}\|^2 = \frac{\|\zeta\|^2}{\min_i \zeta^H H_i \zeta} \leq \frac{3\upsilon_{\mathrm{sdp}}}{\pi/(9m^2)}$.

Thus $$\upsilon_{\mathrm{qp}} \leq \|\hat{z}\|^2 \leq \frac{27}{\pi} m^2 \upsilon_{\mathrm{sdp}}.$$

Take

$$n = 2, \quad |I_i| = 1, \quad h_i = \begin{bmatrix} \cos(\frac{2\pi}{m} i) \\ \sin(\frac{2\pi}{m} i) \end{bmatrix}, \; i = 1, ..., m$$

- For any QP feas. soln z, $\exists\, i$ such $|h_i^H z| \le \frac{\pi}{m}\|z\| \;\Rightarrow \|z\|^2 \ge \frac{m^2}{\pi^2} \;\Rightarrow$ $v_{\mathrm{qp}} \ge \frac{m^2}{\pi^2}$
- $Z = I$ is a feas. soln of SDP, so $v_{\mathrm{sdp}} \le \mathrm{Tr}(I) = 2$

Thus

$$v_{\mathrm{qp}} \ge \frac{1}{2\pi^2} m^2 \, v_{\mathrm{sdp}}$$

$\mathcal{H} = \mathbb{C}^n$

Proof of upper bound is similar to the real case, but with

$$\eta_k \overset{\text{i.i.d.}}{\sim} N_c(0,1) \qquad \left(\text{density } \frac{e^{-|\eta_k|^2}}{\pi}\right)$$

Then $\quad \mathrm{P}(\zeta^H H_i \zeta < \gamma) \leq \frac{4}{3}\gamma \quad \forall\, \gamma > 0,\ \forall\, i$

so

$$\begin{aligned}
& \mathrm{P}\left(\zeta^H H_i \zeta \geq \gamma,\ i = 1, \ldots, m \;\&\; \|\zeta\|^2 \leq \mu \mathrm{Tr}(Z^*)\right) \\
\geq\ & 1 - \sum_{i=1}^{m} \mathrm{P}\left(\zeta^H H_i \zeta < \gamma\right) - \mathrm{P}\left(\|\zeta\|^2 > \mu \mathrm{Tr}(Z^*)\right) \\
\geq\ & 1 - m\frac{4}{3}\gamma - \frac{1}{\mu} \\
>\ & 0 \qquad \text{if } \mu = 2,\ \gamma = \frac{1}{4m}
\end{aligned}$$

Proof of lower bound involves a more intricate example.

Improved approximation bound: bounded phase spread

Theorem 2 (LSTZ '05): $\mathcal{H} = \mathbb{C}^n$. If

- $$h_\ell = \sum_{i=1}^{p} \beta_{i\ell} g_i, \quad \ell = 1, ..., M,$$
  for some $p \geq 1$, $\beta_{i\ell} \in \mathbb{C}$, $g_i \in \mathbb{C}^n$ with $\|g_i\| = 1$ and $g_i^H g_j = 0$ for all $i \neq j$;
- $\beta_{i\ell} = |\beta_{i\ell}| e^{\mathbf{i}\phi_{i\ell}}$ satisfies, for some $0 \leq \phi < \frac{\pi}{2}$,
  $$|\phi_{i\ell} - \phi_{j\ell}| \leq \phi \quad \forall i, j, \ \forall \ell,$$

then
$$v_{\rm qp} \ \leq \ \frac{1}{\cos(\phi)} v_{\rm sdp}.$$

Numerical experience

- For measured VDSL channel data by France Telecom R&D, SDP solution yields nearly doubling of minimum received signal power relative to no precoding.
  $v_{\mathrm{qp}} = v_{\mathrm{sdp}}$ in over 50% of instances. (SDL '05)

- Simulation with randomly generated h_ℓ ($m = M = 8$, $n = 4$) shows that both the mean and the maximum of the upper bound

$$\frac{\|\hat{x}\|^2}{v_{\mathrm{sdp}}}$$

  are lower in the $\mathcal{H} = \mathbb{C}^{\mathrm{n}}$ case (1.14 and 1.8) than the $\mathcal{H} = \mathbb{R}^{\mathrm{n}}$ case (1.17 and 6.2). Thus, SDP solution is better in the complex case not only in the worst case but also on average.

Maximization QP with convex constraints

$$
\begin{aligned}
v_{\rm qp} := \max_{z\in\mathcal{H}} \quad & \|z\|^2 \\
\text{s.t.} \quad & \sum_{\ell\in I_i} |h_\ell^H z|^2 \le 1, \quad i = 1, ..., m,
\end{aligned}
$$

$$
\begin{aligned}
v_{\rm sdp} \;:=\; \max \quad & \mathrm{Tr}(Z) \\
\text{s.t.} \quad & \mathrm{Tr}(H_i Z) \le 1, \quad i = 1, ..., m, \\
& Z \succeq 0.
\end{aligned}
$$

Then

$$
v_{\rm sdp} \;\ge\; v_{\rm qp} \;\overset{?}{\ge}\; C v_{\rm sdp} \qquad (0 < C \le 1)
$$

Approximation upper & lower bounds

Theorem 3 (NRT '99, LSTZ '05): $v_{\mathrm{qp}} \geq C v_{\mathrm{sdp}}$ where

$$O\left(\frac{1}{\ln(m)}\right) \geq C \geq \frac{1}{4\ln(m) + 2\ln(2)} \qquad \text{if} \quad \mathcal{H} = \mathbb{R}^{\mathrm{n}}$$

$$O\left(\frac{1}{\ln(m)}\right) \geq C \geq \frac{1}{6\ln(m) + 4\ln(100)} \qquad \text{if} \quad \mathcal{H} = \mathbb{C}^{\mathrm{n}}$$

Proof uses $\mathrm{P}(\zeta^H H_i \zeta > \gamma) \leq \mathrm{rank}(H_i)\, e^{-\gamma} \quad \forall\, \gamma > 0,\ \forall\, i$

Conclusions & Open Questions

1. For norm minimization on $\mathbb{R}^n$ ($\mathbb{C}^n$) with m concave quadratic constraints, SDP relaxation yields $O(m^2)$ ($O(m)$) approximation.

2. If phase spread of $h_1, ..., h_M$ are bounded by $0 < \phi < \frac{\pi}{2}$, then SDP relaxation yields $O\left(\frac{1}{\cos(\phi)}\right)$ approximation.

3. For norm maximization on $\mathbb{R}^n$ or $\mathbb{C}^n$ with m convex quadratic constraints, SDP relaxation yields $O\left(\frac{1}{\ln(m)}\right)$ approximation.

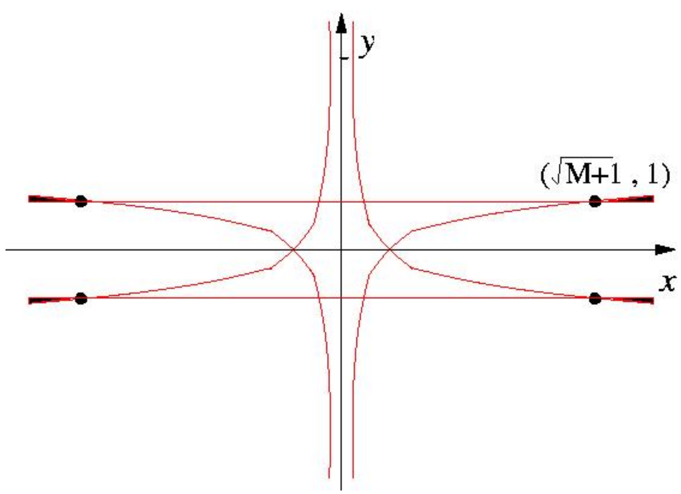
Open Questions:

1. Can the approximation bounds be improved? Adapt SOS relaxation?

2. For nonconcave/convex constraints, SDP relaxation can be arbitrarily bad (for fixed m, n).

$$v_{\rm qp} := \min_{(x,y)\in\mathbb{R}^2} \quad x^2 + y^2$$
$$\text{s.t.} \quad y^2 \geq 1,\ x^2 - Mxy \geq 1,\ x^2 + Mxy \geq 1.$$

$(M > 0)$. Here $v_{\rm qp} = M + 2$ while $v_{\rm sdp} = 2$. Performance of SOS relaxation also worsens with $M \uparrow$. Better approximation?



3. A nonhomogeneous QP:

$$\begin{array}{ll} \min\limits_{z \in \mathcal{H}} & z^H H_0 z + c_0^H z \\ \text{s.t.} & z^H H_i z + c_i^H z \geq 1, \quad i = 1, ..., m, \end{array}$$

can be transformed into a homogeneous QP:

$$\begin{array}{ll} \min\limits_{(z,t) \in \mathcal{H}} & z^H H_0 z + c_0^H z t \\ \text{s.t.} & z^H H_i z + c_i^H z t \geq 1, \quad i = 1, ..., m, \quad t^2 = 1. \end{array}$$

In the case of $m = 2$, $H_1, H_2 \preceq 0$, $c_1 = c_2 = 0$, the approximation bound derived from the SDP relaxation of this homogeneous QP is further improved (from 2 to 1.8) by also using the SDP relaxation of

$$\begin{array}{ll} \min\limits_{z \in \mathcal{H}} & z^H H_0 z \\ \text{s.t.} & z^H H_i z \geq 1, \quad i = 1, ..., m. \end{array}$$

Can this idea be extended?