On SDP and ESDP Relaxation of Sensor Network Localization

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(joint work with Ting Kei Pong)

Talk Outline

- Sensor network localization
- SDP, ESDP relaxations: properties and accuracy certificate

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- SDP, ESDP relaxations: properties and accuracy certificate
- A robust version of ESDP to handle noises
- Log-barrier penalty CGD method
- Numerical simulations
- Conclusion & Ongoing work

Sensor Network Localization

Basic Problem:

- n pts in $\Re^2$.
- Know last $n - m$ pts ('anchors') $x_{m+1}, ..., x_n$ and Eucl. dist. estimate for pairs of 'neighboring' pts

$$d_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A}$$

with $\mathcal{A} \subseteq \{(i,j) : 1 \leq i, j \leq n\}$.

- Estimate first m pts ('sensors').

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History? Graph realization/rigidty, Euclidean matrix completion, position estimation in wireless sensor network, ...

Optimization Problem Formulation

$$\boxed{\; v_{\text{opt}} \;:=\; \min_{x_1,\ldots,x_m} \; \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right| \;}$$

Optimization Problem Formulation

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- Objective function is nonconvex. m can be large ($m \geq 1000$).
- Problem is NP-hard (reduction from PARTITION).
- Local improvement heuristics can fail badly.

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- Objective function is nonconvex. m can be large ($m \geq 1000$).
- Problem is NP-hard (reduction from PARTITION).
- Local improvement heuristics can fail badly.

- Use a convex (SDP, SOCP) relaxation (& local improvement). Low soln accuracy OK. Distributed computation?

SDP Relaxation

Let $X := [x_1 \ \cdots \ x_m]$. $\quad Y = X^T X \quad \Longleftrightarrow \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0, \ \mathrm{rank} Z = 2$

SDP Relaxation

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SDP relaxation (Biswas,Ye '03):

$$\begin{aligned} \upsilon_{\mathrm{sdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}, i\le m<j} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\ & + \sum_{(i,j)\in\mathcal{A}, i<j\le m} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\ & \text{s.t.} \quad Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0 \end{aligned}$$

Adding the nonconvex constraint $\mathrm{rank} Z = 2$ yields original problem.

But SDP relaxation is still expensive to solve for m large..

ESDP Relaxation

ESDP relaxation (Wang, Zheng, Boyd, Ye '06):

$$
\begin{aligned}
\upsilon_{\text{esdp}} := \quad \min_{Z} \quad & \sum_{(i,j)\in\mathcal{A},\, i\le m<j} \left| y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2 \right| \\
+ & \sum_{(i,j)\in\mathcal{A},\, i<j\le m} \left| y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2 \right| \\
\text{s.t.} \quad & Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \\
& \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A},\, i<j\le m
\end{aligned}
$$

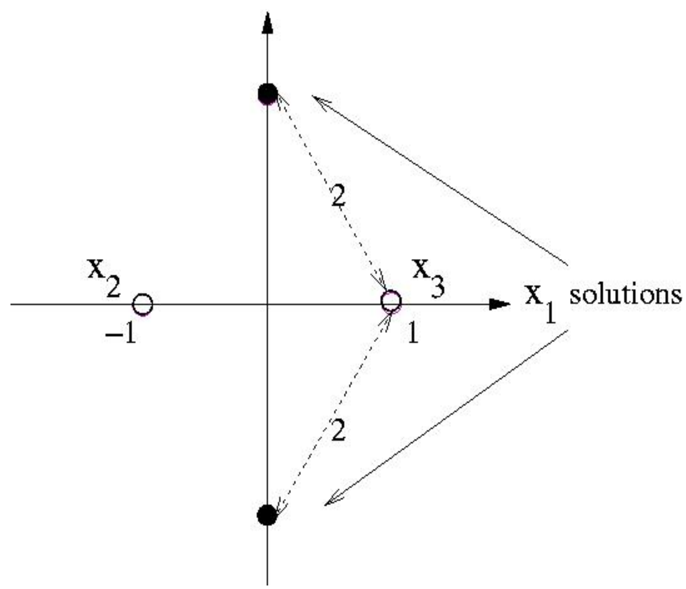
$0 \le \upsilon_{\text{esdp}} \le \upsilon_{\text{sdp}} \le \upsilon_{\text{opt}}$. In simulation, ESDP is nearly as strong as SDP, and solvable much faster by IP method.

Example 1

$n = 3$, $m = 1$, $d_{12} = d_{13} = 2$

Problem:

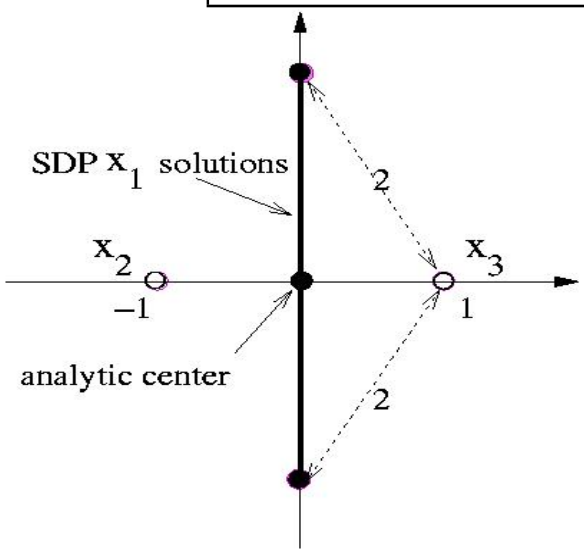
$$0 = \min_{x_1 \in \Re^2} \quad \left| \left\| x_1 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 - 4 \right| + \left| \left\| x_1 - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\|^2 - 4 \right|$$



SDP/ESDP Relaxation:

$$0 = \min_{\substack{x_1=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\in\Re^2 \\ y_{11}\in\Re}} |y_{11} - 2\alpha - 3| + |y_{11} + 2\alpha - 3|$$

$$\text{s.t.} \quad \begin{bmatrix} y_{11} & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \succeq 0$$



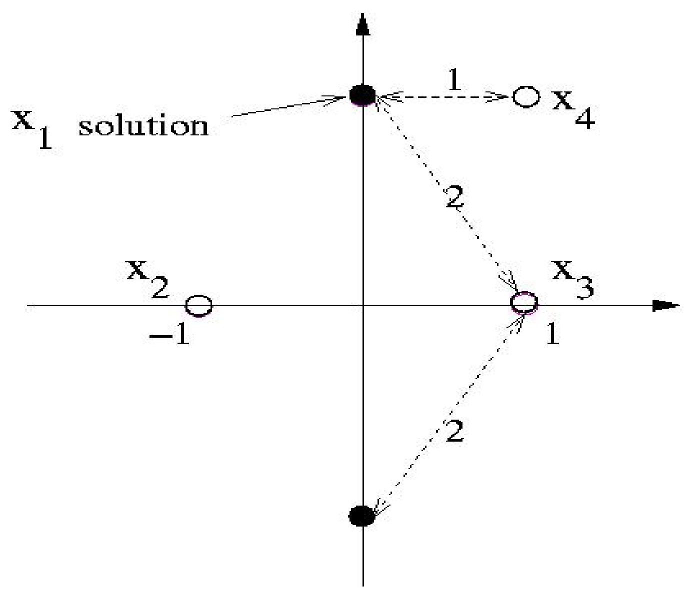
If solve SDP/ESDP by IP method, then likely get analy. center $y_{11} = 3$, $x_1 = \begin{bmatrix}0\\0\end{bmatrix}$

Example 2

$n = 4$, $m = 1$, $d_{12} = d_{13} = 2$, $d_{14} = 1$

Problem:

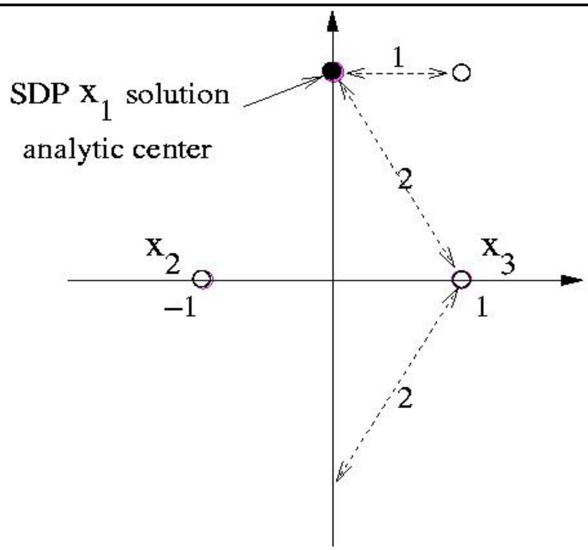
$$0 = \min_{x_1 \in \Re^2} \quad \left| \left\| x_1 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 - 4 \right| + \left| \left\| x_1 - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\|^2 - 4 \right| + \left| \left\| x_1 - \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right\|^2 - 1 \right|$$



SDP/ESDP Relaxation:

$$0 = \min_{\substack{x_1 = \left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] \in \Re^2 \\ y_{11} \in \Re}} |y_{11} - 2\alpha - 3| + |y_{11} + 2\alpha - 3| + |y_{11} - 2\alpha - 2\sqrt{3}\beta + 3|$$

$$\text{s.t.} \quad \begin{bmatrix} y_{11} & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \succeq 0$$



SDP/ESDP has unique soln $y_{11} = 3$, $x_1 = \left[\begin{smallmatrix} 0 \\ \sqrt{3} \end{smallmatrix}\right]$

Properties of SDP & ESDP Relaxations

Assume each $i \leq m$ is conn. to some $j > m$ in the graph $(\{1, ..., n\}, \mathcal{A})$.

Fact 0:

- $\mathrm{Sol(SDP)}$ and $\mathrm{Sol(ESDP)}$ are nonempty, closed, convex.
- If

$$d_{ij} = \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\| \ \ \forall\, (i,j) \in \mathcal{A}$$

"noiseless case"

($x_i^{\mathrm{true}} = x_i \ \forall\, i > m$), then

$$v_{\mathrm{opt}} = v_{\mathrm{sdp}} = v_{\mathrm{esdp}} = 0$$

and

$$Z^{\mathrm{true}} := \begin{bmatrix} X^{\mathrm{true}} & I \end{bmatrix}^T \begin{bmatrix} X^{\mathrm{true}} & I \end{bmatrix}$$

is a soln of SDP and ESDP (i.e., $Z^{\mathrm{true}} \in \mathrm{Sol(SDP)} \subseteq \mathrm{Sol(ESDP)}$).

Let $$\mathrm{tr}_i[Z] := y_{ii} - \|x_i\|^2, \quad i = 1, ..., m.$$ "ith trace"

Fact 1 (Biswas,Ye '03, T '07, Wang et al '06): For each i,

$$\mathrm{tr}_i[Z] = 0 \ \exists Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \quad \Longrightarrow \quad \begin{array}{l} x_i \text{ is invariant over } \mathrm{Sol}(\mathrm{ESDP}) \\ (\text{so } \ x_i = x_i^{\mathrm{true}} \text{ in noiseless case}) \end{array}$$

Still true with "ESDP" changed to "SDP".

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Still true with "ESDP" changed to "SDP".

Fact 2 (Pong, T '09): Suppose $v_{\text{opt}} = 0$. For each i,

$$\mathrm{tr}_i[Z] = 0 \ \forall Z \in \mathrm{Sol(ESDP)} \quad \Longleftarrow \quad x_i \text{ is invariant over Sol(ESDP).}$$

Proof is by induction, starting from sensors that neighbor anchors.
(Q: True for SDP?)

Proof idea:

- If $(i,j) \in \mathcal{A}$ and x_i, x_j are invar. over $\mathrm{Sol(ESDP)}$, then $\mathrm{tr}_i[Z] = \mathrm{tr}_j[Z]$ $\forall Z \in \mathrm{Sol(ESDP)}$.

- Suppose $\exists i \le m$ such that x_i is invar. over $\mathrm{Sol(ESDP)}$ but $\mathrm{tr}_i[\bar{Z}] > 0$ for some $\bar{Z} \in \mathrm{Sol(ESDP)}$. Consider maximal $\bar{\mathcal{I}} \subset \{1, \ldots, m\}$ such that x_i is invar. over $\mathrm{Sol(ESDP)}$ and $\mathrm{tr}_i[\bar{Z}] > 0 \; \forall i \in \bar{\mathcal{I}}$.

- Then x_i is not invar. over $\mathrm{Sol(ESDP)}$ $\forall i \in \mathcal{N}(\bar{\mathcal{I}})$.
  So $\exists Z \in \mathrm{ri(Sol(ESDP))}$ with $x_i \ne \bar{x}_i$ $\forall i \in \mathcal{N}(\bar{\mathcal{I}})$.

- Let $Z^\alpha = \alpha \bar{Z} + (1-\alpha) Z$ with $\alpha > 0$ suff. small.
  Can rotate x_i^α $\forall i \in \bar{\mathcal{I}}$ and Z^α still remains in $\mathrm{Sol(ESDP)}$. $\Rightarrow\Leftarrow$

In practice, there are measurement noises:

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A}.$$

When $\delta := (\delta_{ij})_{(i,j)\in\mathcal{A}} \approx 0$, does $\mathrm{tr}_i[Z] = 0$ (with $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))$) imply $x_i \approx x_i^{\text{true}}$?

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Fact 3 (Pong, T '09): For $\delta \approx 0$ and for each i,

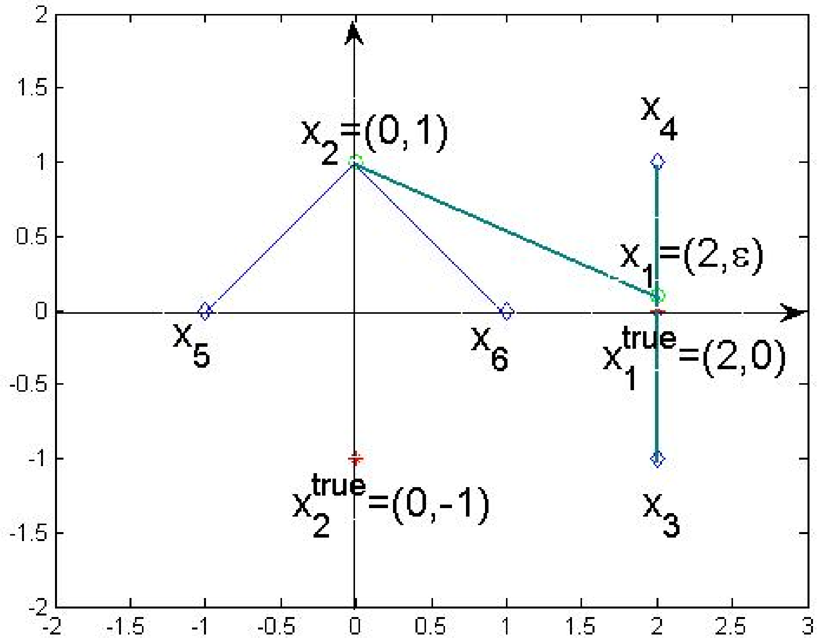
$$\mathrm{tr}_i[Z] = 0 \;\exists Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \quad \not\Longrightarrow \quad x_i \approx x_i^{\text{true}}.$$

Still true with “ESDP” changed to “SDP”.

Proof is by counter-example.

An example of sensitivity of ESDP solns to measurement noise:

Input distance data: $\epsilon > 0$
$d_{12} = \sqrt{4 + (1-\epsilon)^2}, d_{13} = 1+\epsilon, d_{14} = 1-\epsilon, d_{25} = d_{26} = \sqrt{2}$; $m = 2$, $n = 6$.



Thus, even when $Z \in \mathrm{Sol(ESDP)}$ is unique, $\mathrm{tr}_i[Z] = 0$ fails to certify accuracy of x_i in the noisy case!

Robust ESDP

Fix any $\rho_{ij} > |\delta_{ij}|\ \forall (i,j) \in \mathcal{A}$ ($\rho > |\delta|$).

Let $\mathrm{Sol}(\rho\mathrm{ESDP})$ denote the set of $Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix}$ satisfying

$$\boxed{\begin{array}{rl} |y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2| \le \rho_{ij} & \forall (i,j) \in \mathcal{A}, i \le m < j \\ |y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2| \le \rho_{ij} & \forall (i,j) \in \mathcal{A}, i < j \le m \\ \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 & \forall (i,j) \in \mathcal{A}, i < j \le m \end{array}}$$

Note: $Z^{\mathrm{true}} = \begin{bmatrix} X^{\mathrm{true}} & I \end{bmatrix}^T \begin{bmatrix} X^{\mathrm{true}} & I \end{bmatrix} \in \mathrm{Sol}(\rho\mathrm{ESDP})$.

Let

$$Z^{\rho,\delta} := \arg\min_{Z \in \mathrm{Sol}(\rho\mathrm{ESDP})} \sum_{(i,j)\in\mathcal{A}, i<j\le m} - \ln\det \begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}$$

Let

$$Z^{\rho,\delta} := \operatorname*{arg\,min}_{Z\in\mathrm{Sol}(\rho\mathrm{ESDP})} \sum_{(i,j)\in\mathcal{A},i<j\le m} -\ln\det\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}$$

Fact 4 (Pong, T '09): $\exists\ \eta > 0$ and $\bar{\rho} > 0$ such that for each i,

$$\begin{aligned} \mathrm{tr}_i[Z^{\rho,\delta}] < \eta \;\; \exists|\delta| < \rho \le \bar{\rho}e \quad &\Longrightarrow \quad \lim_{|\delta|<\rho\to 0} x_i^{\rho,\delta} = x_i^{\mathrm{true}} \\ \mathrm{tr}_i[Z^{\rho,\delta}] > \tfrac{\eta}{10} \;\; \exists|\delta| < \rho \le \bar{\rho}e \quad &\Longrightarrow \quad x_i \text{ not invar. over } \mathrm{Sol}(\mathrm{ESDP}) \text{ when } \delta = 0 \end{aligned}$$

Moreover,

$$\|x_i^{\rho,\delta} - x_i^{\mathrm{true}}\| \le \sqrt{2|\mathcal{A}| + m}\sqrt{\mathrm{tr}_i[Z^{\rho,\delta}]} \quad \forall\ |\delta| < \rho.$$

Log-barrier Penalty CGD Method

Efficiently compute $Z^{\rho,\delta}$? Let

$$h_a(t) := \frac{1}{2}(t-a)_+^2 + \frac{1}{2}(-t-a)_+^2$$

($|t| \le a \iff h_a(t) = 0$) and

$$\boxed{\begin{aligned} f_\mu(Z) \quad := & \sum_{(i,j)\in\mathcal{A}, i\le m<j} h_{\rho_{ij}}(y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2) \\ & + \sum_{(i,j)\in\mathcal{A}, i<j\le m} h_{\rho_{ij}}(y_{ii} - 2y_{ij} + y_{jj} - d_{ij}^2) \\ & +\mu \sum_{(i,j)\in\mathcal{A}, i<j\le m} -\ln\det\begin{bmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \end{aligned}}$$

- f_μ is partially separable, strictly convex & diff. on its domain.
- For each fixed $\rho > |\delta|$, $\mathrm{argmin} f_\mu \to Z^{\rho,\delta}$ as $\mu \to 0$.

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- For each fixed $\rho > |\delta|$, $\mathrm{argmin} f_\mu \rightarrow Z^{\rho,\delta}$ as $\mu \rightarrow 0$.

Idea: Minimize f_μ approx. by block-coordinate gradient descent (BCGD). (T, Yun '06)

Log-barrier Penalty CGD Method:

Given Z in $\mathrm{dom} f_\mu$, compute gradient $\nabla_{Z_i} f_\mu$ of f_μ w.r.t. $Z_i := \{x_i, y_{ii}, y_{ij} : \ (i,j) \in \mathcal{A}\}$ for each i.

- If $\|\nabla_{Z_i} f_\mu\| \geq \max\{\mu, 10^{-7}\}$ for some i, update Z_i by moving along the Newton direction $-\left(\partial^2_{Z_i Z_i} f_\mu\right)^{-1} \nabla_{Z_i} f_\mu$ with Armijo stepsize rule.

- Decrease μ when $\|\nabla_{Z_i} f_\mu\| < \max\{\mu, 10^{-6}\} \ \ \forall\, i$.

$\mu_{\mathrm{initial}} = 10$, $\mu_{\mathrm{final}} = 10^{-14}$. Decrease μ by a factor of 10 each time.

Coded in Fortran. Compute Newton direc. by sparse Cholesky.
Computation easily distributes.

Simulation Results

- Compare ρESDP as solved by LPCGD method with ESDP as solved by Sedumi 1.05 Sturm (with the interface to Sedumi coded by Wang et al).

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- Anchors and sensors $x_1^{\text{true}}, ..., x_n^{\text{true}}$ uniformly distributed in $[-.5, .5]^2$, $m = .9n$. $(i, j) \in \mathcal{A}$ whenever $\|x_i^{\text{true}} - x_j^{\text{true}}\| < rr$. Set

$$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \cdot |1 + \sigma \cdot \epsilon_{ij}|,$$

where $\epsilon_{ij} \sim N(0, 1)$.

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where $\epsilon_{ij} \sim N(0, 1)$.

- Sensor i is judged as "accurately positioned" if

$$\text{tr}_i[Z^{\text{found}}] < (.01 + 30\sigma) d_{ij}^{\text{avg}}.$$

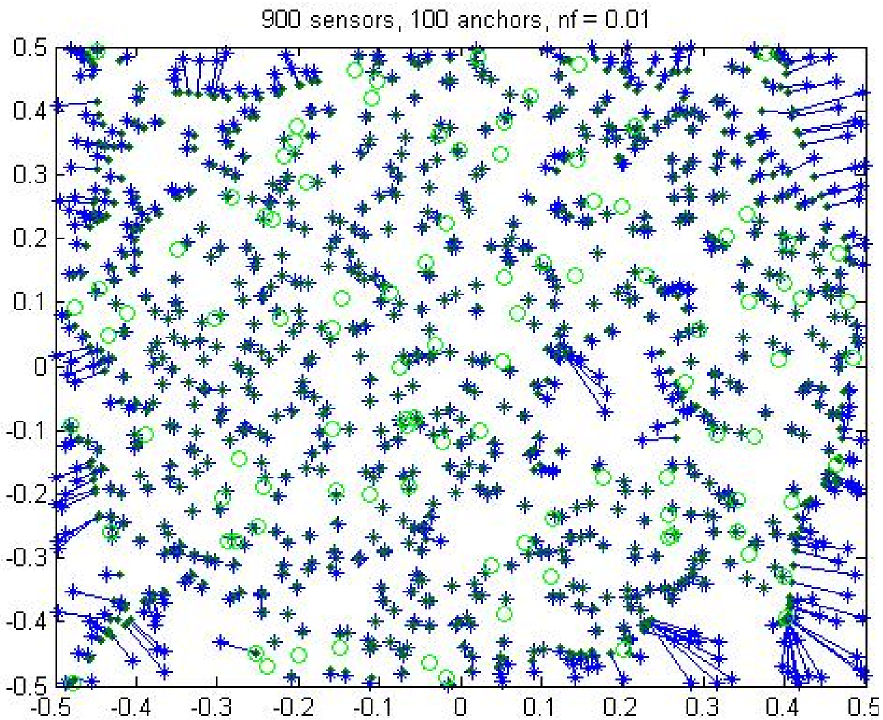
				ρ**ESDP**$_{\text{LPCGD}}$	**ESDP**$_{\text{Sedumi}}$
n	m	σ	rr	**cpu/**m_{ap}**/**err_{ap}	**cpu(cpus)/**m_{ap}**/**err_{ap}
1000	900	0	.06	7/662/1.7e-3	182(104)/669/2.1e-3
1000	900	.01	.06	5/660/2.2e-2	119(42)/720/3.1e-2
2000	1800	0	.06	26/1762/3.1e-4	1157(397)/1742/3.9e-4
2000	1800	.01	.06	20/1699/1.4e-2	966(233)/1746/2.4e-2
10000	9000	0	.02	77/7844/2.3e-3	16411(1297)/6481/2.5e-3
10000	9000	.01	.02	63/8336/1.0e-2	16368(1264)/8593/8.7e-3

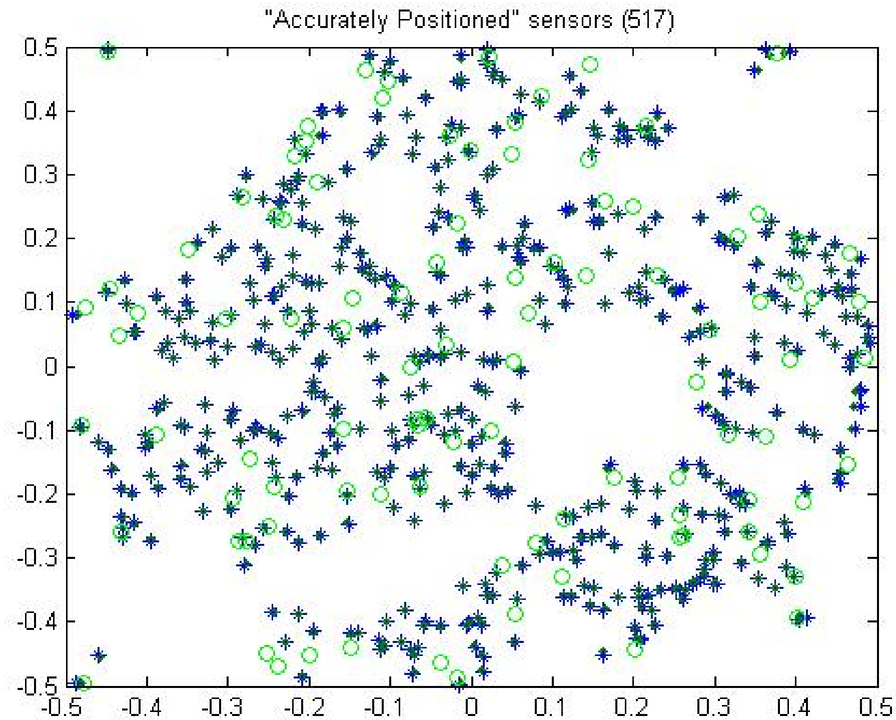
- cpu(sec) times are on a HP DL360 workstation, running Linux 3.5. ESDP is solved by Sedumi; cpus:$=$ run time for Sedumi.

- Set $\rho_{ij} = d_{ij}^2 \cdot ((1-2\sigma)^{-2} - 1)$.

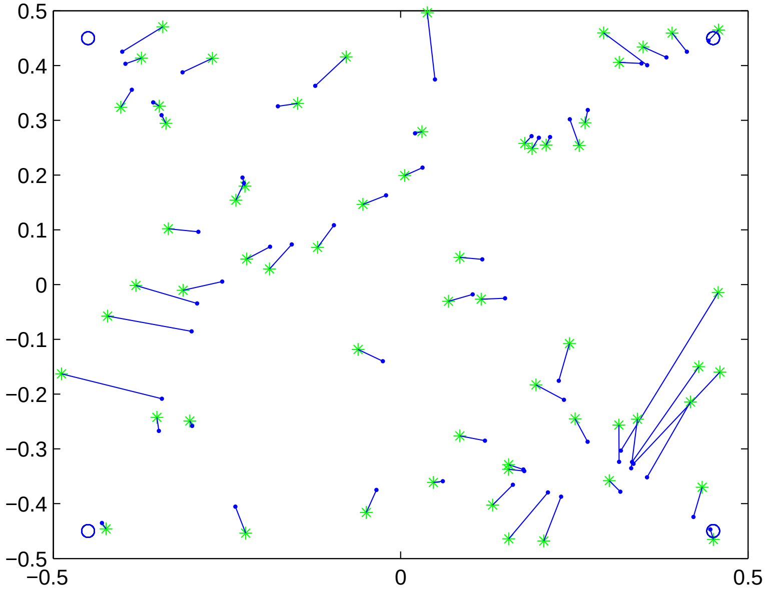
- m_{ap} := # accurately positioned sensors.
  $err_{ap} := \max_{i \text{ accurate. pos.}} \|x_i - x_i^{\text{true}}\|$.

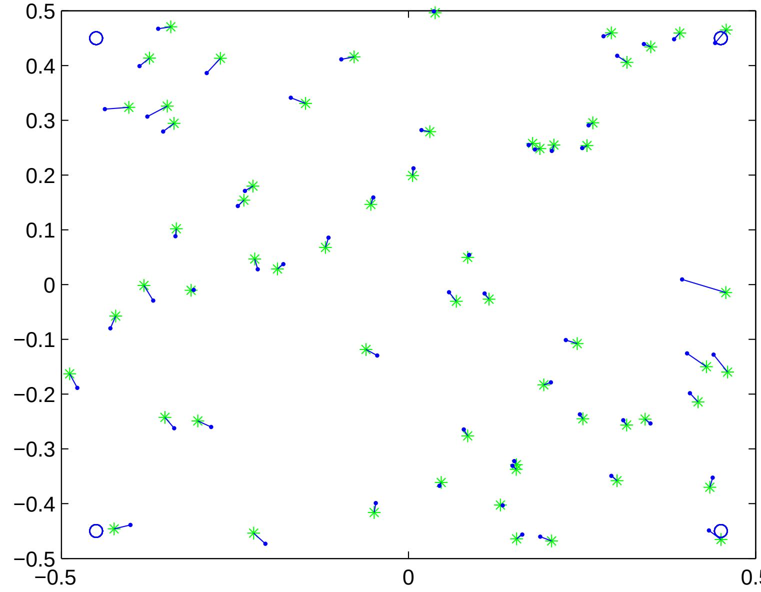
900 sensors, 100 anchors, $rr = 0.06$, $\sigma = 0.01$, solve ρESDP by LPCGD method. x_i^{true} (shown as $*$) and $x_i^{\rho,\delta}$ (shown as ●) are joined by blue line segment; anchors are shown as ○.





60 sensors, 4 anchors at corners, $rr = 0.3$, $\sigma = 0.1$. x_i^{true} (shown as $*$) and $x_i^{\rho,\delta}$ (shown as •) are joined by blue line segment; anchors are shown as ∘. Left: Soln of ρESDP found by LPCGD method. Right: After local gradient improvement.





Conclusion & Ongoing work

- SDP and ESDP solns are sensitive to measurement noise. Lack soln accuracy certificate (though the trace test works well enough in simulation).

- ρESDP has more stable solns. Has soln accuracy certificate (which works well enough in simulation). Needs to estimate the noise level δ to set ρ. Can $\rho > |\delta|$ be relaxed?

- SDP, ESDP, ρESDP solns can be further refined by local improvement. This improves the rmsd when noise level is high (10%).

- Approximation bounds? Extension to maxmin dispersion problem?

Thanks for coming!