On ESDP Relaxation of Sensor Network Localization

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(Ongoing work with Ting Kei Pong)

Talk Outline

- Sensor network localization and SDP, ESDP relaxations
- Properties of SDP, ESDP
- A robust version of ESDP for the noisy case
- Conclusion & Ongoing work

Sensor Network Localization

Basic Problem:

- n pts in $\Re^2$.
- Know last $n-m$ pts ('anchors') $x_{m+1}, ..., x_n$ and Eucl. dist. estimate for pairs of 'neighboring' pts

$$d_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A}$$

  with $\mathcal{A} \subseteq \{(i,j) : 1 \leq i < j \leq n\}$.

- Estimate first m pts ('sensors').

History? Graph realization, position estimation in wireless sensor network, ...

Optimization Problem Formulation

$$v_{\text{opt}} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|_2^2 - d_{ij}^2 \right|$$

- Objective function is nonconvex. m can be large ($m > 1000$).
- Problem is NP-hard (reduction from PARTITION).
- Use a convex (SDP, SOCP) relaxation. High soln accuracy unnecessary.
- Seek “simple” distributed methods (important for practical implementation).

SDP Relaxation

Let $X := [x_1 \ \cdots \ x_m], \quad A := [x_{m+1} \ \cdots \ x_n].$

SDP relaxation (Biswas,Ye '03):

$$
\begin{aligned}
v_{\text{sdp}} := \quad \min_{Z} & \sum_{(i,j)\in\mathcal{A}, j>m} \left| Y_{ii} - 2x_j^T x_i + \|x_j\|_2^2 - d_{ij}^2 \right| \\
+ & \sum_{(i,j)\in\mathcal{A}, j\le m} \left| Y_{ii} - 2Y_{ij} + Y_{jj} - d_{ij}^2 \right| \\
\text{s.t.} \quad & Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \succeq 0
\end{aligned}
$$

Adding the nonconvex constraint $\mathrm{rank} Z = 2$ yields original problem.
$v_{\text{sdp}} \le v_{\text{opt}}$.

But SDP relaxation is still expensive to solve for m large..

SOCP Relaxation

$$v_{\rm opt} = \min_{x_1,\ldots,x_m,y_{ij}} \sum_{(i,j)\in\mathcal{A}} \left|y_{ij} - d_{ij}^2\right|$$
$$\text{s.t.} \quad y_{ij} = \|x_i - x_j\|_2^2 \quad \forall (i,j)\in\mathcal{A}$$

Relax "$=$" to "$\geq$" constraint (Doherty,Pister,El Ghaoui '03):

$$v_{\rm socp} := \min_{x_1,\ldots,x_m,y_{ij}} \sum_{(i,j)\in\mathcal{A}} \left|y_{ij} - d_{ij}^2\right|$$
$$\text{s.t.} \quad y_{ij} \geq \|x_i - x_j\|_2^2 \quad \forall (i,j)\in\mathcal{A}$$

$v_{\rm socp} \leq v_{\rm sdp}$.

SOCP is much easier to solve than SDP relaxation (T '07), but can be much weaker.

ESDP Relaxation

ESDP relaxation (Wang, Zheng, Boyd, Ye '06):

$$
\begin{aligned}
\upsilon_{\text{esdp}} := \quad \min_{Z} & \sum_{(i,j)\in\mathcal{A}, j>m} \left| Y_{ii} - 2x_j^T x_i + \|x_j\|_2^2 - d_{ij}^2 \right| \\
& + \sum_{(i,j)\in\mathcal{A}, j\le m} \left| Y_{ii} - 2Y_{ij} + Y_{jj} - d_{ij}^2 \right| \\
\text{s.t.} \quad & Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix} \\
& \begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}, j \le m \\
& \begin{bmatrix} Y_{ii} & x_i^T \\ x_i & I \end{bmatrix} \succeq 0 \qquad \forall i \le m
\end{aligned}
$$

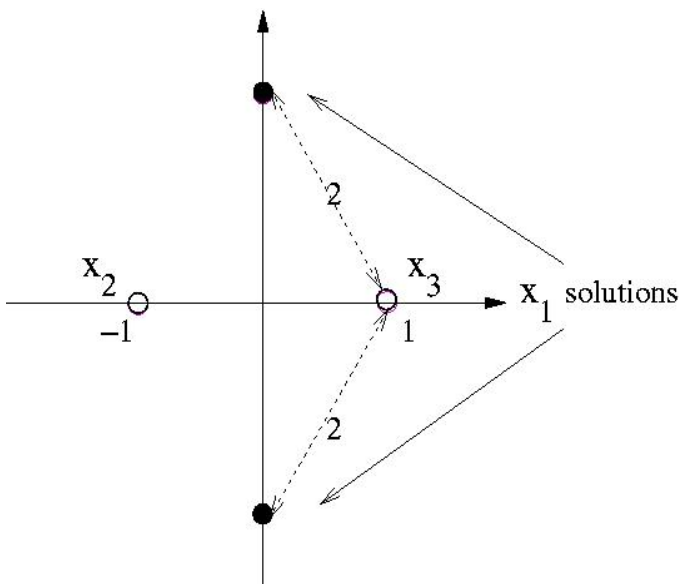
$\upsilon_{\text{socp}} \le \upsilon_{\text{esdp}} \le \upsilon_{\text{sdp}}$. In simulation, ESDP is nearly as strong as SDP, and solvable much faster by IP method.

An Example

$n = 3$, $m = 1$, $d_{12} = d_{13} = 2$

Problem:

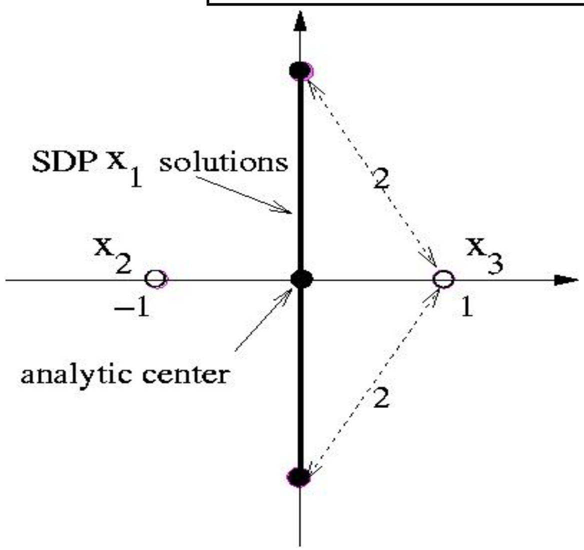
$$0 = \min_{x_1 \in \Re^2} \quad |\|x_1 - (1,0)\|_2^2 - 4| + |\|x_1 - (-1,0)\|_2^2 - 4|$$



SDP/ESDP Relaxation:

$$0 = \min_{\substack{x_1=(\alpha,\beta)\in\Re^2 \\ Y_{11}\in\Re}} |Y_{11} - 2\alpha - 3| + |Y_{11} + 2\alpha - 3|$$

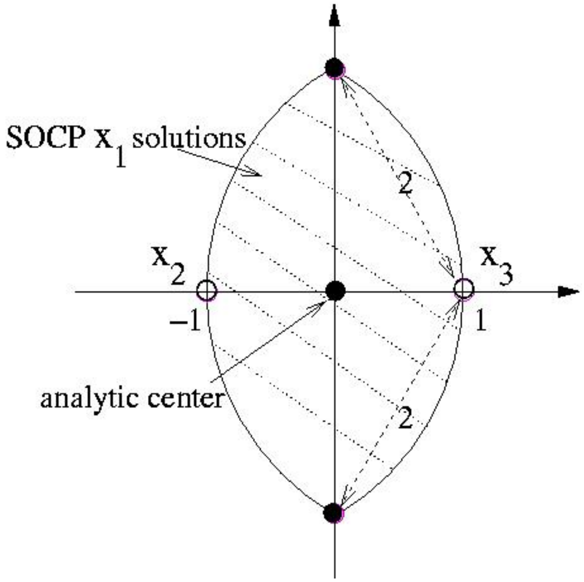
$$\text{s.t.} \quad \begin{bmatrix} Y_{11} & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \succeq 0$$



If solve SDP/ESDP by IP method, then likely get analy. center.

SOCP Relaxation:

$$0 = \min_{\substack{x_1 \in \Re^2 \\ y_{12}, y_{13} \in \Re}} |y_{12} - 4| + |y_{13} - 4|$$
$$\text{s.t.} \quad y_{12} \geq \|x_1 - (1,0)\|_2^2$$
$$y_{13} \geq \|x_1 - (-1,0)\|_2^2$$



If solve SOCP by IP method, then likely get analy. center.

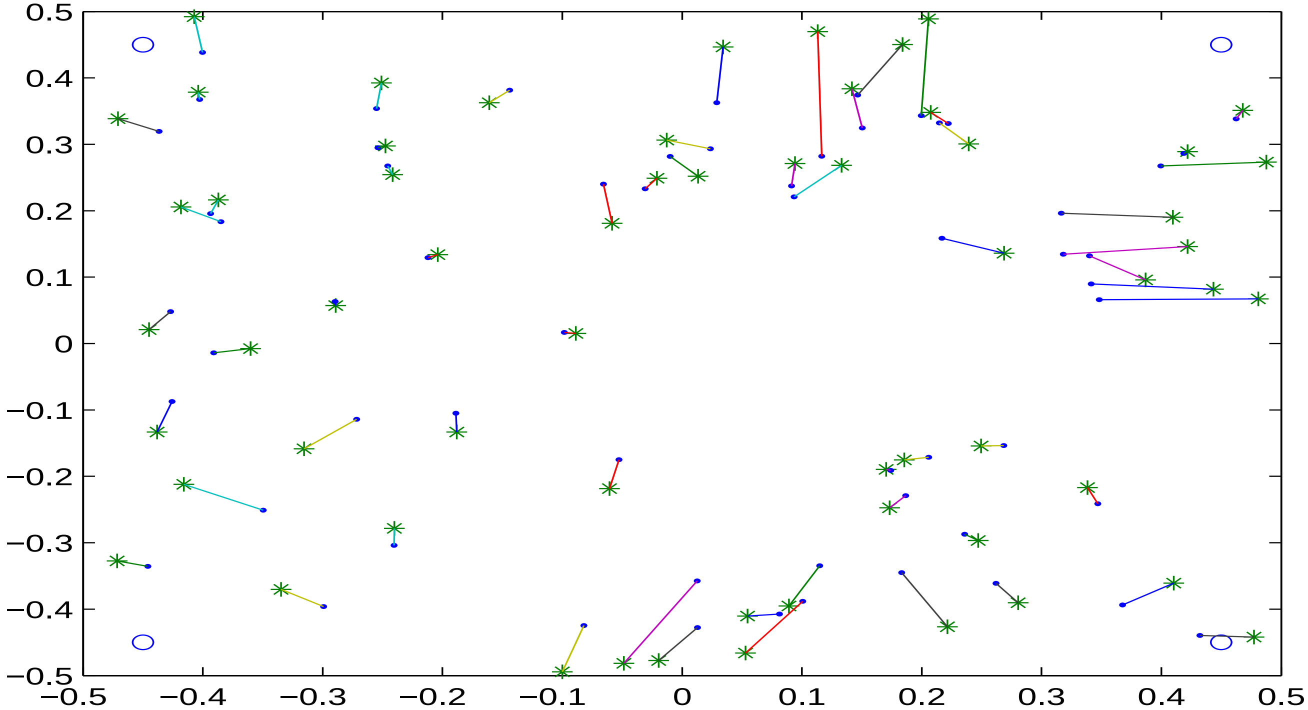
SDP Relaxation: a larger example with noise:

$n = 64$, $m = 60$. Anchors at $(\underline{+}.45, \underline{+}.45)$ ("$\circ$"). Sensors uniformly distributed on $[-.5, .5]^2$ ("$*$").

$(i, j) \in \mathcal{A}$ whenever $\|x_i^{\text{true}} - x_j^{\text{true}}\|_2 \leq 0.3$

Normally distributed noise: $d_{ij} = d_{ij}^{\text{true}} \cdot \max\{0, 1 + .2\nu\}$, $\nu \sim N(0, 1)$.

The SDP soln found by SeDuMi 1.05 is shown ("$\cdot$") joined to its true position ("$*$") by a line.


0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5

Properties of SDP & ESDP Relaxations

Fact 0: Sol(SDP) and Sol(ESDP) are nonempty, closed, convex, and bounded if each $i \le m$ is conn. to some $j > m$ in the graph $(\{1, ..., n\}, \mathcal{A})$.

$$\mathrm{tr}_i[Z] := Y_{ii} - \|x_i\|_2^2, \quad i = 1, ..., m.$$ "ith trace"

Fact 1 (Biswas,Ye '03, T '07, Wang et al '06): For each i,

$$\mathrm{tr}_i[Z] = 0 \; \exists Z \in \mathrm{ri}(\mathrm{Sol(ESDP)}) \quad \Longrightarrow \quad x_i \text{ is invariant over Sol(ESDP)}.$$

Still true with "ESDP" changed to "SDP".

Fact 2 (Pong, T '08): Suppose $\upsilon_{\mathrm{opt}} = 0$. For each i,

$$\mathrm{tr}_i[Z] = 0 \; \forall Z \in \mathrm{Sol(ESDP)} \quad \Longleftrightarrow \quad x_i \text{ is invariant over Sol(ESDP)}.$$

Proof is by induction, starting from sensors that neighbor anchors.
(Q: True for SDP?)

Proof sketch for Fact 2:

1. For $(i,j) \in \mathcal{A}$, $j > m$, if x_i is invariant over Sol(ESDP), then $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathrm{Sol(ESDP)}$.

Why: $\upsilon_{\mathrm{opt}} = 0$ and x_i invariant over Sol(ESDP) imply, for any $Z \in \mathrm{Sol(ESDP)}$,

$$Y_{ii} - 2x_j^T x_i + \|x_j\|_2^2 = d_{ij}^2, \qquad \|x_i - x_j\|_2^2 = d_{ij}^2$$

So $\mathrm{tr}_i(Z) = Y_{ii} - \|x_i\|_2^2 = d_{ij}^2 - \|x_i - x_j\|_2^2 = 0.$

2. For $(i,j) \in \mathcal{A}$, $j \le m$, if x_i is invariant over Sol(ESDP), then $\mathrm{tr}_i(Z) = \mathrm{tr}_j(Z)$ for all $Z \in \mathrm{Sol(ESDP)}$.

Why? $\upsilon_{\mathrm{opt}} = 0$ and x_i invariant over Sol(ESDP) imply, for any $Z \in \mathrm{Sol(ESDP)}$,

$$Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij}^2, \qquad \|x_i - x_j\|_2^2 = d_{ij}^2$$

So $Y_{ij} - x_i^T x_j = \frac{1}{2}(\mathrm{tr}_i(Z) + \mathrm{tr}_j(Z)).$

Then

$$\begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} = \operatorname{tr}_i(Z) \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \operatorname{tr}_j(Z) \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} x_i^T \\ x_j^T \\ I \end{bmatrix} \begin{bmatrix} x_i & x_j & I \end{bmatrix}$$

This is psd, which implies ...that $\operatorname{tr}_i(Z) = \operatorname{tr}_j(Z)$.

When there is measurement noise, does $\mathrm{tr}_i[Z] = 0$ (with $Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP}))$) imply x_i is near the true position of sensor i?

Let

$$d_{ij}^2 = \bar{d}_{ij}^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A},$$

where $\bar{d}_{ij} := \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\|_2$ ($x_i^{\mathrm{true}} = x_i \; \forall \, i > m$). $\qquad \bar{\delta} := \max_{(i,j)\in\mathcal{A}} |\delta_{ij}|$.

Fact 3 (Pong, T '08): For $\bar{\delta} \approx 0$ and for each i,

$$\mathrm{tr}_i[Z] = 0 \; \exists Z \in \mathrm{ri}(\mathrm{Sol}(\mathrm{ESDP})) \quad \not\Longrightarrow \quad \|x_i - x_i^{\mathrm{true}}\|_2 \approx 0.$$

Still true with “ESDP” changed to “SDP”.

Proof is by counter-example.

An example of sensitivity of SDP/ESDP solns to measurement noise:

Thus, even when $Z \in \mathrm{Sol}(\mathrm{SDP}/\mathrm{ESDP})$ is unique, $\mathrm{tr}_i[Z] = 0$ certifies accuracy of x_i only in the noiseless case!

Robust ESDP

Fix $\rho > \bar{\delta}$.

$\mathrm{Sol}(\rho\mathrm{ESDP})$ denotes the set of $Z = \begin{bmatrix} Y & X^T \\ X & I \end{bmatrix}$ satisfying

$$\begin{aligned} \begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix} &\succeq 0 \quad \forall (i,j) \in \mathcal{A}, j \le m \\ \begin{bmatrix} Y_{ii} & x_i^T \\ x_i & I \end{bmatrix} &\succeq 0 \quad \forall i \le m \\ |Y_{ii} - 2x_j^T x_i + \|x_j\|^2 - d_{ij}^2| &\le \rho \quad \forall (i,j) \in \mathcal{A}, j > m \\ |Y_{ii} - 2Y_{ij} + Y_{jj} - d_{ij}^2| &\le \rho \quad \forall (i,j) \in \mathcal{A}, j \le m \end{aligned}$$

Note: $\begin{bmatrix} X^{\mathrm{true}} & I \end{bmatrix}^T \begin{bmatrix} X^{\mathrm{true}} & I \end{bmatrix} \in \mathrm{Sol}(\rho\mathrm{ESDP})$.

Let

$$Z^\rho := \arg\min_{Z \in \mathrm{Sol}(\rho\mathrm{ESDP})} \; - \sum_{(i,j)\in\mathcal{A}, j\le m} \ln\det\left(\begin{bmatrix} Y_{ii} & Y_{ij} & x_i^T \\ Y_{ij} & Y_{jj} & x_j^T \\ x_i & x_j & I \end{bmatrix}\right) - \sum_{i\le m} \ln\det\left(\begin{bmatrix} Y_{ii} & x_i^T \\ x_i & I \end{bmatrix}\right)$$

Fact 4 (Pong, T '08): $\exists \bar\rho > \bar\delta$ and $\tau > 0$ such that, for $\bar\delta < \rho \le \bar\rho$ and for each i,

$$x_i \text{ is invariant over } \mathrm{Sol}(\mathrm{ESDP}|_{\bar{\mathrm{d}}_{\mathrm{ij}}}) \iff \mathrm{tr}_i[Z^\rho] < \tau$$

$$\implies \|x_i^\rho - x_i^{\mathrm{true}}\|_2 \le \sqrt{2|\mathcal{A}| + n}\,(\mathrm{tr}_i[Z^\rho])^{1/2}$$

Conclusion & Ongoing work

SDP and ESDP are stronger relaxations, but inherit the soln instability relative to measurement noise. Lack soln accuracy certificate.

SOCP and ρESDP are weaker relaxations, but have more stable solns. Have soln accuracy certificate. Is ρESDP better?

- Distributed method to compute Z^ρ?
- Simulation and numerical testing?

Thanks for coming!