Chapter 2. Poisson Processes

Prof. Ai-Chun Pang  
Graduate Institute of Networking and Multimedia,  
Department of Computer Science and Information Engineering,  
National Taiwan University, Taiwan
Outline

• Introduction to Poisson Processes
• Properties of Poisson processes
  – Inter-arrival time distribution
  – Waiting time distribution
  – Superposition and decomposition
• Non-homogeneous Poisson processes (relaxing stationary)
• Compound Poisson processes (relaxing single arrival)
• Modulated Poisson processes (relaxing independent)
• Poisson Arrival See Average (PASTA)
(i) $n_{th}$ arrival epoch $\tilde{S}_n$ is
\[
\tilde{S}_n = \tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_n = \sum_{i=1}^{n} \tilde{x}_i
\]
\[
\tilde{S}_0 = 0
\]
(ii) Number of arrivals at time $t$ is: $\tilde{n}(t)$. Notice that:
\[
\{\tilde{n}(t) \geq n\} \iff \{\tilde{S}_n \leq t\}, \{\tilde{n}(t) = n\} \iff \{\tilde{S}_n \leq t \text{ and } \tilde{S}_{n+1} > t\}
**Introduction**

**Arrival Process:** \( X = \{\tilde{x}_i, i = 1, 2, \ldots\} \); \( \tilde{x}_i \)'s can be any
\[ S = \{\tilde{S}_i, i = 0, 1, 2, \ldots\} \); \( \tilde{S}_i \)'s can be any
\[ N = \{\tilde{n}(t), t \geq 0\} \); \( \longrightarrow \) called arrival process

**Renewal Process:** \( X = \{\tilde{x}_i, i = 1, 2, \ldots\} \); \( \tilde{x}_i \)'s are i.i.d.
\[ S = \{\tilde{S}_i, i = 0, 1, 2, \ldots\} \); \( \tilde{S}_i \)'s are general distributed
\[ N = \{\tilde{n}(t), t \geq 0\} \); \( \longrightarrow \) called renewal process

**Poisson Process:** \( X = \{\tilde{x}_i, i = 1, 2, \ldots\} \); \( \tilde{x}_i \)'s are iid exponential distributed
\[ S = \{\tilde{S}_i, i = 0, 1, 2, \ldots\} \); \( \tilde{S}_i \)'s are Erlang distributed
\[ N = \{\tilde{n}(t), t \geq 0\} \); \( \longrightarrow \) called Poisson process
Counting Processes

- A stochastic process \( N = \{ \tilde{n}(t), t \geq 0 \} \) is said to be a \textit{counting process} if \( \tilde{n}(t) \) represents the total number of “events” that have occurred up to time \( t \).

- From the definition we see that for a counting process \( \tilde{n}(t) \) must satisfy:
  1. \( \tilde{n}(t) \geq 0 \).
  2. \( \tilde{n}(t) \) is integer valued.
  3. If \( s < t \), then \( \tilde{n}(s) \leq \tilde{n}(t) \).
  4. For \( s < t \), \( \tilde{n}(t) - \tilde{n}(s) \) equals the number of events that have occurred in the interval \( (s, t] \).
Definition 1: Poisson Processes

The counting process \( N = \{ \tilde{n}(t), t \geq 0 \} \) is a *Poisson process* with rate \( \lambda \) (\( \lambda > 0 \)), if:

1. \( \tilde{n}(0) = 0 \)

2. Independent increments \( \Rightarrow \) *Modulated Poisson Process*

\[
P[\tilde{n}(t) - \tilde{n}(s) = k_1 | \tilde{n}(r) = k_2, r \leq s < t] = P[\tilde{n}(t) - \tilde{n}(s) = k_1]
\]

3. Stationary increments \( \Rightarrow \) *Non-homogeneous Poisson Process*

\[
P[\tilde{n}(t + s) - \tilde{n}(t) = k] = P[\tilde{n}(l + s) - \tilde{n}(l) = k]
\]

4. Single arrival \( \Rightarrow \) *Compound Poisson Process*

\[
P[\tilde{n}(h) = 1] = \lambda h + o(h)
\]

\[
P[\tilde{n}(h) \geq 2] = o(h)
\]

Prof. Ai-Chun Pang, NTU
Definition 2: Poisson Processes

The counting process \( N = \{\tilde{n}(t), \ t \geq 0\} \) is a *Poisson process* with rate \( \lambda \) (\( \lambda > 0 \)), if:

1. \( \tilde{n}(0) = 0 \)

2. Independent increments

3. The number of events in any interval of length \( t \) is Poisson distributed with mean \( \lambda t \). That is, for all \( s, t \geq 0 \)

\[
P[\tilde{n}(t + s) - \tilde{n}(s) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots
\]
Theorem: Definitions 1 and 2 are equivalent.

Proof. We show that Definition 1 implies Definition 2. To start, fix $u \geq 0$ and let

$$g(t) = E[e^{-u\tilde{n}(t)}]$$

We derive a differential equation for $g(t)$ as follows:

$$g(t + h) = E[e^{-u\tilde{n}(t+h)}]$$

$$= E \left\{ e^{-u\tilde{n}(t)} e^{-u[\tilde{n}(t+h) - \tilde{n}(t)]} \right\}$$

$$= E \left[ e^{-u\tilde{n}(t)} \right] E \left\{ e^{-u[\tilde{n}(t+h) - \tilde{n}(t)]} \right\} \quad \text{by independent increments}$$

$$= g(t) E \left[ e^{-u\tilde{n}(h)} \right] \quad \text{by stationary increments} \quad (1)$$
Theorem: Definitions 1 and 2 are equivalent.

Conditioning on whether \( \tilde{n}(t) = 0 \) or \( \tilde{n}(t) = 1 \) or \( \tilde{n}(t) \geq 2 \) yields

\[
E \left[ e^{-u\tilde{n}(h)} \right] = 1 - \lambda h + o(h) + e^{-u}(\lambda h + o(h)) + o(h) \\
= 1 - \lambda h + e^{-u}\lambda h + o(h)
\]

(2)

From (1) and (2), we obtain that

\[
g(t + h) = g(t)(1 - \lambda h + e^{-u}\lambda h) + o(h)
\]

implying that

\[
\frac{g(t + h) - g(t)}{h} = g(t)\lambda(e^{-u} - 1) + \frac{o(h)}{h}
\]
Theorem: Definitions 1 and 2 are equivalent.

Letting $h \to 0$ gives

$$g'(t) = g(t) \lambda(e^{-u} - 1)$$

or, equivalently,

$$\frac{g'(t)}{g(t)} = \lambda(e^{-u} - 1)$$

Integrating, and using $g(0) = 1$, shows that

$$\log(g(t)) = \lambda t(e^{-u} - 1)$$

or

$$g(t) = e^{\lambda t(e^{-u} - 1)} \quad \text{→ the Laplace transform of a Poisson r. v.}$$

Since $g(t)$ is also the Laplace transform of $\tilde{n}(t)$, $\tilde{n}(t)$ is a Poisson r. v.
**Theorem: Definitions 1 and 2 are equivalent**

Let $P_n(t) = P[\tilde{n}(t) = n]$.

We derive a differential equation for $P_0(t)$ in the following manner:

\[
P_0(t + h) = P[\tilde{n}(t + h) = 0] = P[\tilde{n}(t) = 0, \tilde{n}(t + h) - \tilde{n}(t) = 0] = P[\tilde{n}(t) = 0]P[\tilde{n}(t + h) - \tilde{n}(t) = 0] = P_0(t)[1 - \lambda h + o(h)]
\]

Hence

\[
\frac{P_0(t + h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h}
\]

Letting $h \to 0$ yields

\[
P_0'(t) = -\lambda P_0(t)
\]

Since $P_0(0) = 1$, then

\[
P_0(t) = e^{-\lambda t}
\]
Theorem: Definitions 1 and 2 are equivalent

Similarly, for \( n \geq 1 \)

\[
P_n(t + h) = P[\tilde{n}(t + h) = n]
\]
\[
= P[\tilde{n}(t) = n, \tilde{n}(t + h) - \tilde{n}(t) = 0]
\]
\[
+ P[\tilde{n}(t) = n - 1, \tilde{n}(t + h) - \tilde{n}(t) = 1]
\]
\[
+ P[\tilde{n}(t + h) = n, \tilde{n}(t + h) - \tilde{n}(t) \geq 2]
\]
\[
= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h)
\]
\[
= (1 - \lambda h)P_n(t) + \lambda hP_{n-1}(t) + o(h)
\]

Thus

\[
\frac{P_n(t + h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}
\]

Letting \( h \to 0 \),

\[
P'_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)
\]
The Inter-Arrival Time Distribution

Theorem. Poisson Processes have exponential inter-arrival time distribution, i.e., \( \{\tilde{x}_n, n = 1, 2, \ldots\} \) are i.i.d and exponentially distributed with parameter \( \lambda \) (i.e., mean inter-arrival time = \( 1/\lambda \)).

Proof.

\[ \tilde{x}_1 : \quad P(\tilde{x}_1 > t) = P(\tilde{n}(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t} \]

\[ \therefore \tilde{x}_1 \sim e(t; \lambda) \]

\[ \tilde{x}_2 : \quad P(\tilde{x}_2 > t|\tilde{x}_1 = s) \]
\[ = P\{0 \text{ arrivals in } (s, s + t)|\tilde{x}_1 = s\} \]
\[ = P\{0 \text{ arrivals in } (s, s + t]\}(\text{by independent increment}) \]
\[ = P\{0 \text{ arrivals in } (0, t]\}(\text{by stationary increment}) \]
\[ = e^{-\lambda t} \quad \therefore \tilde{x}_2 \text{ is independent of } \tilde{x}_1 \text{ and } \tilde{x}_2 \sim exp(t; \lambda). \]

\[ \Rightarrow \text{ The procedure repeats for the rest of } \tilde{x}_i \text{'s.} \]
The Arrival Time Distribution of the $n$th Event

**Theorem.** The arrival time of the $n_{th}$ event, $\tilde{S}_n$ (also called the waiting time until the $n_{th}$ event), is *Erlang* distributed with parameter $(n, \lambda)$.

**Proof.**  

**Method 1:**

\[
\begin{align*}
\therefore P[\tilde{S}_n \leq t] &= P[\tilde{n}(t) \geq n] = \sum_{k=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \\
\therefore f_{\tilde{S}_n}(t) &= \lambda e^{-\lambda t}(\lambda t)^{n-1} \frac{1}{(n-1)!} \quad \text{(exercise)}
\end{align*}
\]

**Method 2:**

\[
\begin{align*}
f_{\tilde{S}_n}(t)dt &= dF_{\tilde{S}_n}(t) = P[t < \tilde{S}_n < t + dt] \\
&= P\{n - 1 \text{ arrivals in } (0, t] \text{ and 1 arrival in } (t, t + dt)\} + o(dt) \\
&= P[\tilde{n}(t) = n - 1 \text{ and 1 arrival in } (t, t + dt)] + o(dt) \\
&= P[\tilde{n}(t) = n - 1]P[1 \text{ arrival in } (t, t + dt)] + o(dt)(\text{why?})
\end{align*}
\]
The Arrival Time Distribution of the $n$th Event

\[
e^{-\lambda t}(\lambda t)^{n-1} \frac{\lambda dt + o(dt)}{(n-1)!}
\]

\[
\therefore \lim_{dt \to 0} \frac{f_{\tilde{S}_n}(t)dt}{dt} = f_{\tilde{S}_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}
\]
Conditional Distribution of the Arrival Times

**Theorem.** Given that \( \tilde{n}(t) = n \), the \( n \) arrival times \( \tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_n \) have the same distribution as the order statistics corresponding to \( n \) i.i.d. uniformly distributed random variables from \((0, t)\).

**Order Statistics.** Let \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \) be \( n \) i.i.d. continuous random variables having common pdf \( f \). Define \( \tilde{x}_{(k)} \) as the \( k_{th} \) smallest value among all \( \tilde{x}_i \)'s, i.e., \( \tilde{x}_{(1)} \leq \tilde{x}_{(2)} \leq \tilde{x}_{(3)} \leq \ldots \leq \tilde{x}_{(n)} \), then \( \tilde{x}_{(1)}, \ldots, \tilde{x}_{(n)} \) are known as the “order statistics” corresponding to random variables \( \tilde{x}_1, \ldots, \tilde{x}_n \). We have that the joint pdf of \( \tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)} \) is

\[
 f_{\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)}}(x_1, x_2, \ldots, x_n) = n! f(x_1) f(x_2) \ldots f(x_n),
\]

where \( x_1 < x_2 < \ldots < x_n \) (check the textbook [Ross]).
Conditional Distribution of the Arrival Times

Proof. Let $0 < t_1 < t_2 < \ldots < t_{n+1} = t$ and let $h_i$ be small enough so that $t_i + h_i < t_{i+1}$, $i = 1, \ldots, n$.

\[
P[t_i < \tilde{S}_i < t_i + h_i, \ i = 1, \ldots, n|\tilde{n}(t) = n] = \frac{P\left(\begin{array}{c}
\text{exactly one arrival in each } [t_i, t_i + h_i] \\
i = 1, 2, \ldots, n, \text{ and no arrival elsewhere in } [0, t]
\end{array}\right)}{P[\tilde{n}(t) = n]}
\]

\[
= \frac{(e^{-\lambda h_1} \lambda h_1)(e^{-\lambda h_2} \lambda h_2) \ldots (e^{-\lambda h_n} \lambda h_n)(e^{-\lambda(t-h_1-h_2-\ldots-h_n)})}{e^{-\lambda t}(\lambda t)^n/n!}
\]

\[
= \frac{n!(h_1h_2h_3\ldots h_n)}{t^n}
\]

\[
P[t_i < \tilde{S}_i < t_i + h_i, \ i = 1, \ldots, n|\tilde{n}(t) = n] = \frac{n!}{t^n}
\]
Conditional Distribution of the Arrival Times

Taking \( \lim_{h_i \to 0, i=1,...,n} ( ) \), then

\[
f_{\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n | \tilde{n}(t)}(t_1, t_2, \ldots, t_n | n) = \frac{n!}{t_n}, \quad 0 < t_1 < t_2 < \ldots < t_n.
\]
Conditional Distribution of the Arrival Times

**Example** (see Ref [Ross], Ex. 2.3(A) p.68). Suppose that travellers arrive at a train depot in accordance with a Poisson process with rate $\lambda$. If the train departs at time $t$, what is the expected sum of the waiting times of travellers arriving in $(0, t)$? That is, $E[\sum_{i=1}^{\tilde{n}(t)}(t - \tilde{S}_i)] = ?$
**Conditional Distribution of the Arrival Times**

**Answer.** Conditioning on $\tilde{n}(t) = n$ yields

$$E[\sum_{i=1}^{n} (t - \tilde{S}_i)|\tilde{n}(t) = n] = nt - E[\sum_{i=1}^{n} \tilde{S}_i]$$

$$= nt - E[\sum_{i=1}^{n} \tilde{u}(i)] \quad \text{(by the theorem)}$$

$$= nt - E[\sum_{i=1}^{n} \tilde{u}_i] \quad \therefore \sum_{i=1}^{n} \tilde{u}(i) = \sum_{i=1}^{n} \tilde{u}_i$$

$$= nt - \frac{t}{2} \cdot n = \frac{nt}{2} \quad \therefore E[\tilde{u}_i] = \frac{t}{2}$$

To find $E[\sum_{i=1}^{\tilde{n}(t)} (t - \tilde{S}_i)]$, we should take another expectation

\[ \therefore E[\sum_{i=1}^{\tilde{n}(t)} (t - \tilde{S}_i)] = \frac{t}{2} \cdot E[\tilde{n}(t)] = \frac{\lambda t^2}{2} \]
Superposition of Independent Poisson Processes

**Theorem.** Superposition of independent Poisson Processes 
$$(\lambda_i, i = 1, \ldots, N),$$ is also a Poisson process with rate $\sum_{i=1}^{N} \lambda_i$. 

**Homework** Prove the theorem (note that a Poisson process must satisfy Definitions 1 or 2).
Decomposition of a Poisson Process

Theorem.

• Given a Poisson process \( N = \{\tilde{n}(t), t \geq 0\} \);

• If \( \tilde{n}_i(t) \) represents the number of type-\( i \) events that occur by time \( t, i = 1, 2 \);

• Arrival occurring at time \( s \) is a type-1 arrival with probability \( p(s) \), and type-2 arrival with probability \( 1 - p(s) \)

\[ \downarrow \] then

• \( \tilde{n}_1, \tilde{n}_2 \) are independent,

• \( \tilde{n}_1(t) \sim P(k; \lambda tp) \), and

• \( \tilde{n}_2(t) \sim P(k; \lambda t(1 - p)) \), where \( p = \frac{1}{t} \int_0^t p(s)ds \)
Decomposition of a Poisson Process

If \( p(s) = p \) is constant, then

\[
P(\lambda) \rightarrow \text{Poisson rate } \lambda p
\]

\[
P(1-p) \rightarrow \text{Poisson rate } \lambda(1-p)
\]
Decomposition of a Poisson Process

Proof. It is to prove that, for fixed time $t$, 

$$P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m] = P[\tilde{n}_1(t) = n]P[\tilde{n}_2(t) = m]$$

$$= \frac{e^{-\lambda pt} (\lambda pt)^n}{n!} \cdot \frac{e^{-\lambda(1-p)t} [\lambda (1-p)t]^m}{m!}$$

.................................

$$P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m]$$

$$= \sum_{k=0}^{\infty} P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m|\tilde{n}_1(t) + \tilde{n}_2(t) = k] \cdot P[\tilde{n}_1(t) + \tilde{n}_2(t) = k]$$

$$= P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m|\tilde{n}_1(t) + \tilde{n}_2(t) = n + m] \cdot P[\tilde{n}_1(t) + \tilde{n}_2(t) = n + m]$$
Decomposition of a Poisson Process

- From the “condition distribution of the arrival times”, any event occurs at some time that is uniformly distributed, and is independent of other events.

- Consider that only one arrival occurs in the interval \([0, t]\):

\[
P[\text{type - 1 arrival}|\tilde{n}(t) = 1] = \int_0^t P[\text{type - 1 arrival}|\text{arrival time } \tilde{S}_1 = s, \tilde{n}(t) = 1] \times f_{\tilde{S}_1|\tilde{n}(t)}(s|\tilde{n}(t) = 1)ds
\]

\[
= \int_0^t P(s) \cdot \frac{1}{t} ds = \frac{1}{t} \int_0^t P(s) ds = p
\]
Decomposition of a Poisson Process

\[ \begin{align*}
\therefore & \quad P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m] \\
& = P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m | \tilde{n}_1(t) + \tilde{n}_2(t) = n + m] \cdot P[\tilde{n}_1(t) + \tilde{n}_2(t) = n + m] \\
& = \binom{n + m}{n} p^n (1 - p)^m \cdot \frac{e^{-\lambda t} (\lambda t)^{n+m}}{(n + m)!} \\
& = \frac{(n + m)!}{n! m!} p^n (1 - p)^m \cdot \frac{e^{-\lambda t} (\lambda t)^{n+m}}{(n + m)!} \\
& = \frac{e^{-\lambda pt} (\lambda pt)^n}{n!} \cdot \frac{e^{-\lambda (1-p)t} [\lambda (1 - p)t]^m}{m!}
\end{align*} \]
Decomposition of a Poisson Process

**Example** (An Infinite Server Queue, textbook [Ross]).

- \( G_{\tilde{S}}(t) = P(\tilde{S} \leq t) \), where \( \tilde{S} = \) service time
- \( G_{\tilde{S}}(t) \) is independent of each other and of the arrival process
- \( \tilde{n}_1(t) \): the number of customers which have left before \( t \);
- \( \tilde{n}_2(t) \): the number of customers which are still in the system at time \( t \);

\[ \Rightarrow \tilde{n}_1(t) \sim ? \text{ and } \tilde{n}_2(t) \sim ? \]
Decomposition of a Poisson Process

Answer.

$\tilde{n}_1(t)$: the number of type-1 customers

$\tilde{n}_2(t)$: the number of type-2 customers

\begin{align*}
\text{type-1: } P(s) &= P(\text{finish before } t) \\
&= P(\tilde{S} \leq t - s) = G_{\tilde{s}}(t - s) \\
\text{type-2: } 1 - P(s) &= \bar{G}_{\tilde{s}}(t - s)
\end{align*}

\[
\therefore \quad \tilde{n}_1(t) \sim P \left( k; \lambda t \cdot \frac{1}{t} \int_0^t G_{\tilde{s}}(t - s) ds \right) \\
\tilde{n}_2(t) \sim P \left( k; \lambda t \cdot \frac{1}{t} \int_0^t \bar{G}_{\tilde{s}}(t - s) ds \right)
\]
Decomposition of a Poisson Process

\[ E[\tilde{n}_1(t)] = \lambda t \cdot \frac{1}{t} \int_0^t G(t - s) ds \]

\[ = \lambda \int_t^0 G(y)(-dy) \]

\[ = \lambda \int_0^t G(y) dy \]

As \( t \to \infty \), we have

\[ \lim_{t \to \infty} E[\tilde{n}_2(t)] = \lambda \int_0^t \bar{G}(y) dy = \lambda E[\bar{S}] \quad \text{(Little’s formula)} \]
Non-homogeneous Poisson Processes

- The counting process $N = \{\tilde{n}(t), t \geq 0\}$ is said to be a non-stationary or non-homogeneous Poisson Process with time-varying intensity function $\lambda(t), t \geq 0$, if:
  
  1. $\tilde{n}(0) = 0$
  2. $N$ has independent increments
  3. $P[\tilde{n}(t + h) - \tilde{n}(t) \geq 2] = o(h)$
  4. $P[\tilde{n}(t + h) - \tilde{n}(t) = 1] = \lambda(t) \cdot h + o(h)$

- Define “integrated intensity function” $m(t) = \int_0^t \lambda(t')dt'$.

**Theorem.**

\[
P[\tilde{n}(t + s) - \tilde{n}(t) = n] = \frac{e^{-[m(t+s)-m(t)]}[m(t+s) - m(t)]^n}{n!}
\]

**Proof.** < Homework >.
Non-homogeneous Poisson Processes

Example. The “output process” of the $M/G/\infty$ queue is a non-homogeneous Poisson process having intensity function $\lambda(t) = \lambda G(t)$, where $G$ is the service distribution.

Hint. Let $D(s, s + r)$ denote the number of service completions in the interval $(s, s + r]$ in $(0, t]$. If we can show that

- $D(s, s + r)$ follows a Poisson distribution with mean $\lambda \int_s^{s+r} G(y)dy$, and
- the numbers of service completions in disjoint intervals are independent,

then we are finished by definition of a non-homogeneous Poisson process.
Non-homogeneous Poisson Processes

Answer.

• An arrival at time $y$ is called a type-1 arrival if its service completion occurs in $(s, s + r]$.

• Consider three cases to find the probability $P(y)$ that an arrival at time $y$ is a type-1 arrival:

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$s + r$</td>
<td>$t$</td>
</tr>
</tbody>
</table>

– Case 1: $y \leq s$.

$$P(y) = P\{s - y < \tilde{S} < s + r - y\} = G(s + r - y) - G(s - y)$$

– Case 2: $s < y \leq s + r$.

$$P(y) = P\{\tilde{S} < s + r - y\} = G(s + r - y)$$
Non-homogeneous Poisson Processes

- Case 3: $s + r < y \leq t$.

\[ P(y) = 0 \]

- Based on the decomposition property of a Poisson process, we conclude that $D(s, s + r)$ follows a Poisson distribution with mean $\lambda pt$, where $p = (1/t) \int_0^t P(y)dy$.

\[
\int_0^t P(y)dy = \int_0^s [G(s + r - y) - G(s - y)]dy + \int_s^{s+r} G(s + r - y)dy \\
+ \int_{s+r}^t (0)dy \\
= \int_0^{s+r} G(s + r - y)dy - \int_0^s G(s - y)dy \\
= \int_0^{s+r} G(z)dz - \int_0^s G(z)dz = \int_s^{s+r} G(z)dz
\]
Non-homogeneous Poisson Processes

• Because of

  – the independent increment assumption of the Poisson arrival process, and
  – the fact that there are always servers available for arrivals,

⇒ the departure process has independent increments
Compound Poisson Processes

- A stochastic process \( \{\tilde{x}(t), t \geq 0\} \) is said to be a *compound Poisson process* if
  - it can be represented as
    \[
    \tilde{x}(t) = \sum_{i=1}^{\tilde{n}(t)} \tilde{y}_i, \quad t \geq 0
    \]
  - \( \{\tilde{n}(t), t \geq 0\} \) is a Poisson process
  - \( \{\tilde{y}_i, i \geq 1\} \) is a family of independent and identically distributed random variables which are also independent of \( \{\tilde{n}(t), t \geq 0\} \)

- The random variable \( \tilde{x}(t) \) is said to be a compound Poisson random variable.

- \( E[\tilde{x}(t)] = \lambda t E[\tilde{y}_i] \) and \( Var[\tilde{x}(t)] = \lambda t E[\tilde{y}_i^2] \).
Compound Poisson Processes

- **Example** (Batch Arrival Process). Consider a parallel-processing system where each job arrival consists of a possibly random number of tasks. Then we can model the arrival process as a compound Poisson process, which is also called a *batch arrival process*.

- Let $\tilde{y}_i$ be a random variable that denotes the number of tasks comprising a job. We derive the probability generating function $P_{\tilde{x}}(t)(z)$ as follows:

$$
P_{\tilde{x}}(t)(z) = E \left[ z^{\tilde{x}(t)} \right] = E \left[ E \left[ z^{\tilde{x}(t)} | \tilde{n}(t) \right] \right] = E \left[ E \left[ z^{\tilde{y}_1 + \cdots + \tilde{y}_{\tilde{n}}(t)} | \tilde{n}(t) \right] \right] 
$$

$$
= E \left[ E \left[ z^{\tilde{y}_1 + \cdots + \tilde{y}_{\tilde{n}}(t)} \right] \right] \quad \text{(by independence of $\tilde{n}(t)$ and $\{\tilde{y}_i\}$)}
$$

$$
= E \left[ E \left[ z^{\tilde{y}_1} \right] \cdots E \left[ z^{\tilde{y}_{\tilde{n}}(t)} \right] \right] \quad \text{(by independence of $\tilde{y}_1, \cdots, \tilde{y}_{\tilde{n}}(t)$)}
$$

$$
= E \left[ (P_{\tilde{y}}(z))^{\tilde{n}(t)} \right] = P_{\tilde{n}(t)} (P_{\tilde{y}}(z))
$$
Modulated Poisson Processes

- Assume that there are two states, 0 and 1, for a “modulating process.”

\[
\begin{array}{c}
0 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\end{array}
\]

- When the state of the modulating process equals 0 then the arrive rate of customers is given by \( \lambda_0 \), and when it equals 1 then the arrival rate is \( \lambda_1 \).

- The residence time in a particular modulating state is exponentially distributed with parameter \( \mu \) and, after expiration of this time, the modulating process changes state.

- The initial state of the modulating process is randomly selected and is equally likely to be state 0 or 1.
Modulated Poisson Processes

- For a given period of time $(0, t)$, let $\Upsilon$ be a random variable that indicates the total amount of time that the modulating process has been in state 0. Let $\tilde{x}(t)$ be the number of arrivals in $(0, t)$.

- Then, given $\Upsilon$, the value of $\tilde{x}(t)$ is distributed as a non-homogeneous Poisson process and thus

  $$P[\tilde{x}(t) = n | \Upsilon = \tau] = \frac{(\lambda_0 \tau + \lambda_1 (t - \tau)) n e^{-(\lambda_0 \tau + \lambda_1 (t - \tau))}}{n!}$$

- As $\mu \to 0$, the probability that the modulating process makes no transitions within $t$ seconds converges to 1, and we expect for this case that

  $$P[\tilde{x}(t) = n] = \frac{1}{2} \left\{ \frac{(\lambda_0 t)^n e^{-\lambda_0 t}}{n!} + \frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!} \right\}$$
Modulated Poisson Processes

- As $\mu \to \infty$, then the modulating process makes an infinite number of transitions within $t$ seconds, and we expect for this case that
  
  $$P[\tilde{x}(t) = n] = \frac{(\beta t)^n e^{-\beta t}}{n!}, \quad \text{where } \beta = \frac{\lambda_0 + \lambda_1}{2}$$

- **Example** (Modeling Voice).
  - A basic feature of speech is that it comprises an alternation of silent periods and non-silent periods.
  - The arrival rate of packets during a talk spurt period is Poisson with rate $\lambda_1$ and silent periods produce a Poisson rate with $\lambda_0 \approx 0$.
  - The duration of times for talk and silent periods are exponentially distributed with parameters $\mu_1$ and $\mu_0$, respectively.

$\Rightarrow$ The model of the arrival stream of packets is given by a modulated Poisson process.
Poisson Arrivals See Time Averages (PASTA)

- PASTA says: as \( t \to \infty \)

  Fraction of arrivals who see the system in a given state upon arrival (arrival average)

  \[ = \]  Fraction of time the system is in a given state (time average)

  \[ = \]  The system is in the given state at any random time after being steady

- Counter-example (textbook [Kao]: Example 2.7.1)
Poisson Arrivals See Time Averages (PASTA)

- Arrival average that an arrival will see an idle system = 1
- Time average of system being idle = 1/2

• Mathematically,
  - Let \( X = \{\tilde{x}(t), t \geq 0\} \) be a stochastic process with state space \( S \), and \( B \subset S \)
  - Define an indicator random variable
    \[
    \tilde{u}(t) = \begin{cases} 
    1, & \text{if } \tilde{x}(t) \in B \\
    0, & \text{otherwise}
    \end{cases}
    \]
  - Let \( N = \{\tilde{n}(t), t \geq 0\} \) be a Poisson process with rate \( \lambda \) denoting the arrival process

then,
Poisson Arrivals See Time Averages (PASTA)

\[
\lim_{t \to \infty} \frac{\int_0^t \tilde{u}(s)d\tilde{n}(s)}{\tilde{n}(t)} = \lim_{t \to \infty} \frac{\int_0^t \tilde{u}(s)ds}{t}
\]

(arrival average) \quad (time average)

- Condition – For PASTA to hold, we need the lack of anticipation assumption (LAA): for each \( t \geq 0 \),
  - the arrival process \( \{\tilde{n}(t + u) - \tilde{n}(t), u \geq 0\} \) is independent of \( \{\tilde{x}(s), 0 \leq s \leq t\} \) and \( \{\tilde{n}(s), 0 \leq s \leq t\} \).

- Application:
  - To find the waiting time distribution of any arriving customer
  - Given: \( P[\text{system is idle}] = 1 - \rho; \ P[\text{system is busy}] = \rho \)
Poisson Arrivals See Time Averages (PASTA)

Case 1: system is idle

Case 2: system is busy

\[ \Rightarrow P(\tilde{w} \leq t) = P(\tilde{w} \leq t | \text{idle}) \cdot P(\text{idle upon arrival}) \]
\[ + \quad P(\tilde{w} \leq t | \text{busy}) \cdot P(\text{busy upon arrival}) \]
Memoryless Property of the Exponential Distribution

- A random variable $\tilde{x}$ is said to be without memory, or *memoryless*, if
  
  $$P[\tilde{x} > s + t | \tilde{x} > t] = P[\tilde{x} > s] \quad \text{for all } s, t \geq 0 \quad (3)$$

- The condition in Equation (3) is equivalent to
  
  $$\frac{P[\tilde{x} > s + t, \tilde{x} > t]}{P[\tilde{x} > t]} = P[\tilde{x} > s]$$

  or
  
  $$P[\tilde{x} > s + t] = P[\tilde{x} > s]P[\tilde{x} > t] \quad (4)$$

- Since Equation (4) is satisfied when $\tilde{x}$ is exponentially distributed (for $e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$), it follows that exponential random variable are memoryless.

- Not only is the exponential distribution “memoryless,” but it is the unique continuous distribution possessing this property.
Comparison of Two Exponential Random Variables

Suppose that $\tilde{x}_1$ and $\tilde{x}_2$ are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$. What is $P[\tilde{x}_1 < \tilde{x}_2]$?

\[
P[\tilde{x}_1 < \tilde{x}_2] = \int_0^\infty P[\tilde{x}_1 < \tilde{x}_2 | \tilde{x}_1 = x] \lambda_1 e^{-\lambda_1 x} dx
\]

\[
= \int_0^\infty P[x < \tilde{x}_2] \lambda_1 e^{-\lambda_1 x} dx
\]

\[
= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx
\]

\[
= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x} dx
\]

\[
= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\]
Minimum of Exponential Random Variables

Suppose that $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n$ are independent exponential random variables, with $\tilde{x}_i$ having rate $\mu_i$, $i = 1, \cdots, n$. It turns out that the smallest of the $\tilde{x}_i$ is exponential with a rate equal to the sum of the $\mu_i$.

$$P[\min(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) > x] = P[\tilde{x}_i > x \text{ for each } i = 1, \cdots, n]$$

$$= \prod_{i=1}^{n} P[\tilde{x}_i > x] \quad \text{(by independence)}$$

$$= \prod_{i=1}^{n} e^{-\mu_i x}$$

$$= \exp \left\{ - \left( \sum_{i=1}^{n} \mu_i \right) x \right\}$$

How about $\max(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$? (exercise)