1. Let $f$ be defined for all real $x$, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real $x$ and $y$. Prove that $f$ is constant.

**Proof:** $|f(x) - f(y)| \leq (x - y)^2$ for all real $x$ and $y$. Fix $y$, $|f(x) - f(y)| \leq |x - y|$. Let $x \to y$, therefore,

$$0 \leq \lim_{x \to y} \frac{f(x) - f(y)}{x - y} \leq \lim_{x \to y} |x - y| = 0$$

It implies that $(f(x) - f(y))/(x - y) \to 0$ as $x \to y$. Hence $f'(y) = 0$, $f = \text{const}$.

2. Suppose $f'(x) > 0$ in $(a, b)$. Prove that $f$ is strictly increasing in $(a, b)$, and let $g$ be its inverse function. Prove that $g$ is differentible, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

**Proof:** For every pair $x > y$ in $(a, b)$, $f(x) - f(y) = f'(c)(x - y)$ where $y < c < x$ by Mean-Value Theorem. Note that $c \in (a, b)$ and $f'(x) > 0$ in $(a, b)$, hence $f'(c) > 0$. $f(x) - f(y) > 0$, $f(x) > f(y)$ if $x > y$, $f$ is strictly increasing in $(a, b)$.

Let $\Delta g = g(x_0 + h) - g(x_0)$. Note that $x_0 = f(g(x_0))$, and thus,

$$(x_0 + h) - x_0 = f(g(x_0 + h)) - f(g(x_0)),$$
\[ h = f(g(x_0) + \Delta g) - f(g(x_0)) = f(g + \Delta g) - f(g). \]

Thus we apply the fundamental lemma of differentiation,

\[ h = \left[f'(g) + \eta(\Delta g)\right] \Delta g, \]

Note that \( f'(g(x)) > 0 \) for all \( x \in (a, b) \) and \( \eta(\Delta g) \to 0 \) as \( h \to 0 \), thus,

\[ \lim_{h \to 0} \Delta g/h = \lim_{h \to 0} \frac{1}{f'(g(x))} \cdot \frac{1}{\frac{\eta(\Delta g)}{h}} = \frac{1}{f'(g(x))}. \]

Thus \( g'(x) = \frac{1}{f'(g(x))}, \ g'(f(x)) = \frac{1}{f'(x)}. \)

3. Suppose \( g \) is a real function on \( R^1 \), with bounded derivative (say \( |g'| \leq M \)). Fix \( \epsilon > 0 \), and define \( f(x) = x + \epsilon g(x) \). Prove that \( f \) is one-to-one if \( \epsilon \) is small enough. (A set of admissible values of \( \epsilon \) can be determined which depends only on \( M \).)

**Proof:** For every \( x < y \), and \( x, y \in R \), we will show that \( f(x) \neq f(y) \).

By using Mean-Value Theorem:

\[ g(x) - g(y) = g'(c)(x - y) \quad \text{where} \quad x < c < y, \]

\[ (x - y) + \epsilon((x) - g(y)) = (\epsilon g'(c) + 1)(x - y), \]

that is,

\[ f(x) - f(y) = (\epsilon g'(c) + 1)(x - y). \quad (*) \]

Since \( |g'(x)| \leq M \), \( -M \leq g'(x) \leq M \) for all \( x \in R \). Thus \( 1 - \epsilon M \leq \epsilon g'(c) + 1 \leq 1 + \epsilon M \), where \( x < c < y \). Take \( c = \frac{1}{2M} \), and \( \epsilon g'(c) + 1 > 0 \) where \( x < c < y \) for all \( x, y \). Take into equation \((*)\), and \( f(x) - f(y) < 0 \) since \( x - y < 0 \), that is, \( f(x) \neq f(y) \), that is, \( f \) is one-to-one (injective).
4. If
\[ C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0, \]
where \( C_0, \ldots, C_n \) are real constants, prove that the equation
\[ C_0 + C_1 x + \ldots + C_{n-1} x^{n-1} + C_n x^n = 0 \]
has at least one real root between 0 and 1.

**Proof:** Let \( f(x) = C_0 x + \ldots + \frac{C_n}{n+1} x^{n+1} \). \( f \) is differentiable in \( R^1 \) and \( f(0) = f(1) = 0 \). Thus, \( f(1) - f(0) = f'(c) \) where \( c \in (0, 1) \) by Mean-Value Theorem. Note that
\[ f'(x) = C_0 + C_1 x + \ldots + C_{n-1} x^{n-1} + C_n x^n. \]
Thus, \( c \in (0, 1) \) is one real root between 0 and 1 of that equation.

5. Suppose \( f \) is defined and differentiable for every \( x > 0 \), and \( f'(x) \to 0 \) as \( x \to +\infty \). Put \( g(x) = f(x + 1) - f(x) \). Prove that \( g(x) \to 0 \) as \( x \to +\infty \).

**Proof:** \( f(x + 1) - f(x) = f'(c)(x + 1 - x) \) where \( x < c < x + 1 \) by Mean-Value Theorem. Thus, \( g(x) = f'(c) \) where \( x < c < x + 1 \), that is,
\[ \lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} f'(c) = \lim_{c \to +\infty} f'(c) = 0. \]

6. Suppose
   (a) \( f \) is continuous for \( x \geq 0 \),
   (b) \( f'(x) \) exists for \( x > 0 \),
   (c) \( f(0) = 0 \),
(d) $f'$ is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that $g$ is monotonically increasing.

**Proof:** Our goal is to show $g'(x) > 0$ for all $x > 0$

$$\iff g'(x) = \frac{x f'(x) - f(x)}{x^2} > 0 \iff f'(x) > \frac{f(x)}{x}.$$

Since $f'(x)$ exists, $f(x) - f(0) = f'(c)(x - 0)$ where $0 < c < x$ by Mean-Value Theorem. $\Rightarrow f'(c) = \frac{f(x)}{x}$ where $0 < c < x$. Since $f'$ is monotonically increasing, $f'(x) > f'(c)$, that is, $f'(x) > \frac{f(x)}{x}$ for all $x > 0$.

7. Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$  

(This holds also for complex functions.)

**Proof:**

$$\frac{f'(t)}{g'(t)} = \lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)} = \lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t)}{g(t)}$$

Surely, this holds also for complex functions.

8. Suppose $f'(x)$ is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying $f$ is uniformly differentiable on $[a, b]$ if $f'$ is continuous on $[a, b]$.) Does this hold for vector-valued functions too?
Proof: Since $f'(x)$ is continuous on a compact set $[a, b]$, $f'(x)$ is uniformly continuous on $[a, b]$. Hence, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f'(t) - f'(x)| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. Thus, $f'(c) = \frac{f(t) - f(x)}{t - x}$ where $c$ between $t$ and $x$ by Mean-Value Theorem. Note that $0 < |c - x| < \delta$ and thus $|f'(c) - f'(x)| < \epsilon$, thus,

$$|\frac{f(t) - f(x)}{t - x} - f'(x)| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

Note: It does not hold for vector-valued functions. If not, take

$$f(x) = (\cos x, \sin x),$$

$[a, b] = [0, 2\pi]$, and $x = 0$. Hence $f'(x) = (-\sin x, \cos x)$. Take any $1 > \epsilon > 0$, there exists $\delta > 0$ such that

$$|\frac{f(t) - f(0)}{t - 0} - f'(0)| < \epsilon$$

whenever $0 < |t| < \delta$ by our hypothesis. With calculating,

$$|\left(\frac{\cos t - 1}{t}, \frac{\sin t}{t}\right) - (0, 1)| < \epsilon$$

$$|\left(\frac{\cos t - 1}{t}, \frac{\sin t}{t} - 1\right)| < \epsilon$$

$$\left(\frac{\cos t - 1}{t}\right)^2 + \left(\frac{\sin t}{t} - 1\right)^2 < \epsilon^2 < \epsilon$$

$$\frac{2}{t^2} + 1 - \frac{2(\cos t + \sin t)}{t} < \epsilon$$

since $1 > \epsilon > 0$. Note that

$$\frac{2}{t^2} + 1 - \frac{4}{t} < \frac{2}{t^2} + 1 - \frac{2(\cos t + \sin t)}{t}$$

But $\frac{2}{t^2} + 1 - \frac{4}{t} \to +\infty$ as $t \to 0$. It contradicts.
9. Let \( f \) be a continuous real function on \( R^1 \), of which it is known that 
\( f'(x) \) exists for all \( x \neq 0 \) and that \( f'(x) \to 0 \) as \( x \to 0 \). Does it follow 
that \( f'(0) \) exists?

**Note:** We prove a more general exercise as following.
Suppose that \( f \) is continuous on an open interval \( I \) containing \( x_0 \), sup-
pose that \( f' \) is defined on \( I \) except possibly at \( x_0 \), and suppose that 
\( f'(x) \to L \) as \( x \to x_0 \). Prove that \( f'(x_0) = L \).

**Proof of the Note:** Using L’Hospital’s rule:
\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} f'(x_0 + h)
\]
By our hypothesis: \( f'(x) \to L \) as \( x \to x_0 \). Thus,
\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = L,
\]
Thus \( f'(x_0) \) exists and 
\[
f'(x_0) = L.
\]

10. Suppose \( f \) and \( g \) are complex differentiable functions on \((0, 1)\), \( f(x) \to 0 \), 
\( g(x) \to 0 \), \( f'(x) \to A \), \( g'(x) \to B \) as \( x \to 0 \), where \( A \) and \( B \) are complex 
numbers, \( B \neq 0 \). Prove that
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.
\]
Compare with Example 5.18. **Hint:**
\[
\frac{f(x)}{g(x)} = \left( \frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)}.
\]
Apply Theorem 5.13 to the real and imaginary parts of \( \frac{f(x)}{x} \) and \( \frac{g(x)}{x} \).
Proof: Write \( f(x) = f_1(x) + i f_2(x) \), where \( f_1(x) \), \( f_2(x) \) are real-valued functions. Thus,
\[
\frac{df(x)}{dx} = \frac{df_1(x)}{dx} + i \frac{df_2(x)}{dx},
\]
Apply L’Hospital’s rule to \( \frac{f_1(x)}{x} \) and \( \frac{f_2(x)}{x} \), we have
\[
\lim_{x \to 0} \frac{f_1(x)}{x} = \lim_{x \to 0} f_1'(x)
\]
\[
\lim_{x \to 0} \frac{f_2(x)}{x} = \lim_{x \to 0} f_2'(x)
\]
Combine \( f_1(x) \) and \( f_2(x) \), we have
\[
\lim_{x \to 0} \frac{f_1(x)}{x} + i \lim_{x \to 0} \frac{f_2(x)}{x} = \lim_{x \to 0} \frac{f_1(x)}{x} + i \lim_{x \to 0} \frac{f_2(x)}{x} = \lim_{x \to 0} \frac{f(x)}{x}
\]
or
\[
\lim_{x \to 0} \frac{f_1(x)}{x} + i \lim_{x \to 0} \frac{f_2(x)}{x} = \lim_{x \to 0} f_1'(x) + i \lim_{x \to 0} f_2'(x) = \lim_{x \to 0} f'(x)
\]
Thus, \( \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} f'(x) \). Similarly, \( \lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} g'(x) \).
Note that \( B \neq 0 \). Thus,
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \left( \frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)}
\]
\[
= (A - A) \frac{1}{B} + A \frac{B}{B} = A
\]
Note: In Theorem 5.13, we know \( g(x) \to +\infty \) as \( x \to 0 \). (\( f(x) = x \), and \( g(x) = x + x^2e^{i\pi/2} \)).

11. Suppose \( f \) is defined in a neighborhood of \( x \), and suppose \( f''(x) \) exists. Show that
\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x)
\]
Show by an example that the limit may exist even if \( f''(x) \) does not.

**Hint:** Use Theorem 5.13.

**Proof:** By using L’Hospital’s rule: (respect to \( h \).)

\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h}
\]

Note that

\[
f''(x) = \frac{1}{2} \left( f''(x) + f''(x) \right)
\]

\[
= \frac{1}{2} \left( \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h} + \lim_{h \to 0} \frac{f'(x - h) - f'(x)}{-h} \right)
\]

\[
= \frac{1}{2} \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{h}
\]

\[
= \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h}
\]

Thus,

\[
\frac{f(x + h) + f(x - h) - 2f(x)}{h^2} \to f''(x)
\]

as \( h \to 0 \). Counter-example: \( f(x) = x|x| \) for all real \( x \).

12. If \( f(x) = |x|^3 \), compute \( f'(x) \), \( f''(x) \) for all real \( x \), and show that \( f^{(3)}(0) \) does not exist.

**Proof:** \( f'(x) = 3|x|^2 \) if \( x \neq 0 \). Consider

\[
\frac{f(h) - f(0)}{h} = \frac{|h|^3}{h}
\]

Note that \( |h|/h \) is bounded and \( |h|^2 \to 0 \) as \( h \to 0 \). Thus,

\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0.
\]

Hence, \( f'(x) = 3|x|^2 \) for all \( x \). Similarly,

\[
f''(x) = 6|x|.
\]
Thus,
\[ \frac{f''(h) - f(0)}{h} = 6 \frac{|h|}{h} \]
Since \( \frac{|h|}{h} = 1 \) if \( h > 0 \) and \( = -1 \) if \( h < 0 \), \( f''(0) \) does not exist.

13. Suppose \( a \) and \( c \) are real numbers, \( c > 0 \), and \( f \) is defined on \([-1, 1]\) by
\[ f(x) = \begin{cases} x^a \sin(x^{-c}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \]
Prove the following statements:

(a) \( f \) is continuous if and only if \( a > 0 \).
(b) \( f'(0) \) exists if and only if \( a > 1 \).
(c) \( f' \) is bounded if and only if \( a \geq 1 + c \).
(d) \( f' \) is continuous if and only if \( a > 1 + c \).
(e) \( f''(0) \) exists if and only if \( a > 2 + c \).
(f) \( f'' \) is bounded if and only if \( a \geq 2 + 2c \).
(g) \( f'' \) is continuous if and only if \( a > 2 + 2c \).

**Proof:** For (a): \( \Rightarrow \) \( f \) is continuous iff for any sequence \( \{x_n\} \to 0 \) with \( x_n \neq 0 \), \( x_n^a \sin x_n^{-c} \to 0 \) as \( n \to \infty \). In particular, take
\[ x_n = \left( \frac{1}{2n\pi + \pi/2} \right) \to 0 \]
and thus \( x_n^a \to 0 \) as \( n \to \infty \). Hence \( a > 0 \). (If not, then \( a = 0 \) or \( a < 0 \). When \( a = 0 \), \( x_n^a = 1 \). When \( a < 0 \), \( x_n^a = 1/x_n^{-a} \to \infty \) as \( n \to \infty \). It contradicts.)

\( \Leftarrow \) \( f \) is continuous on \([-1, 1] - \{0\}\) clearly. Note that
\[ -|x^a| \leq x^a \sin(x^{-c}) \leq |x^a|, \]
and \( |x^a| \to 0 \) as \( x \to 0 \) since \( a > 0 \). Thus \( f \) is continuous at \( x = 0 \). Hence \( f \) is continuous.
For (b): \( f'(0) \) exists iff \( x^{a-1}\sin(x^{-c}) \to 0 \) as \( x \to 0 \). In the previous proof we know that \( f'(0) \) exists if and only if \( a - 1 > 0 \). Also, \( f'(0) = 0 \).

14. Let \( f \) be a differentiable real function defined in \((a, b)\). Prove that \( f \) is convex if and only if \( f' \) is monotonically increasing. Assume next that \( f''(x) \) exist for every \( x \in (a, b) \), and prove that \( f \) is convex if and only if \( f''(x) \geq 0 \) for all \( x \in (a, b) \).

15. Suppose \( a \in R^1, f \) is a twice-differentiable real function on \((a, \infty)\), and \( M_0, M_1, M_2 \) are the least upper bounds of \(|f(x)|, |f'(x)|, |f''(x)|\), respectively, on \((a, \infty)\). Prove that

\[
M_1^2 \leq 4M_0M_2.
\]

Hint: If \( h > 0 \), Taylor’s theorem shows that

\[
f'(x) = \frac{1}{2h} [f(x + 2h) - f(x)] - hf''(\xi)
\]

for some \( \xi \in (x, x + 2h) \). Hence

\[
|f'(x)| \leq hM_2 + \frac{M_0}{h}.
\]

To show that \( M_1^2 = 4M_0M_2 \) can actually happen, take \( a = -1 \), define

\[
f(x) = \begin{cases} 
2x^2 - 1 & (-1 < x < 0), \\
\frac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty),
\end{cases}
\]

and show that \( M_0 = 1, M_1 = 4, M_2 = 4 \). Does \( M_1^2 \leq 4M_0M_2 \) hold for vector-valued functions too?

**Proof:** Suppose \( h > 0 \). By using Taylor’s theorem:

\[
f(x + h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi)
\]
for some $x < \xi < x + 2h$. Thus

$$h|f'(x)| \leq |f(x + h)| + |f(x)| + \frac{h^2}{2}|f''(\xi)|$$

$$h|f'(x)| \leq 2M_0 + \frac{h^2}{2}M_2.$$ 

$$h^2M_2 - 2h|f'(x)| + 4M_0 \geq 0 \quad (*)$$

Since equation $(*)$ holds for all $h > 0$, its determinant must be non-positive.

$$4|f'(x)|^2 - 4M_2(4M_0) \leq 0$$

$$|f'(x)|^2 \leq 4M_0M_2$$

$$(M_1)^2 \leq 2M_0M_2$$

**Note:** There is a similar exercise:

Suppose $f(x)(-\infty < x < +\infty)$ is a twice-differentiable real function, and

$$M_k = \sup_{-\infty < x < +\infty} |f^{(k)}(x)| < +\infty \quad (k = 0, 1, 2).$$

Prove that $M_1^2 \leq 2M_0M_2$.

**Proof of Note:**

$$f(x + h) = f(x) + f'(x)h + \frac{f''(\xi_1)}{2}h^2$$

$(x < \xi_1 < x + h$ or $x > \xi_1 > x + h)$ ............................... (*)

$$f(x - h) = f(x) - f'(x)h + \frac{f''(\xi_2)}{2}h^2$$

$(x - h < \xi_2 < x$ or $x - h > \xi_2 > x)$ ............................... (**) 

(*) minus (**):

$$f(x + h) - f(x - h) = 2f'(x)h + \frac{h^2}{2}(f''(\xi_1) - f''(\xi_2)).$$

11
\[
2h|f'(x)| \leq |2hf'(x)|
\]
\[
2h|f'(x)| \leq |f(x + h)| + |f(x - h)| + \frac{h^2}{2}(|f''(\xi_1)| + |f''(\xi_2)|)
\]
\[
2h|f'(x)| \leq 2M_0 + h^2 M_2
\]
\[
M_2 h^2 - 2|f'(x)|h + 2M_0 \geq 0
\]

Since this equation holds for all \( h \), its determinant must be non-positive:

\[
4|f'(x)|^2 - 4M_2(2M_0) \leq 0,
\]
\[
|f'(x)|^2 \leq 2M_0M_2
\]

Thus
\[
M_1^2 \leq 2M_0M_2
\]

16. Suppose \( f \) is twice-differentiable on \((0, \infty)\), \( f'' \) is bounded on \((0, \infty)\), and \( f(x) \to 0 \) as \( x \to \infty \). Prove that \( f'(x) \to 0 \) as \( x \to \infty \). Hint: Let \( a \to \infty \) in Exercise 15.

**Proof:** Suppose \( a \in (0, \infty) \), and \( M_0, M_1, M_2 \) are the least upper bounds of \( |f(x)|, |f'(x)|, |f''(x)| \) on \((a, \infty)\). Hence, \( M_1^2 \leq 4M_0M_2 \). Let \( a \to \infty, M_0 = \sup |f(x)| \to 0 \). Since \( M_2 \) is bounded, therefore \( M_1^2 \to 0 \) as \( a \to \infty \). It implies that \( \sup |f'(x)| \to 0 \) as \( x \to \infty \).

17. Suppose \( f \) is a real, three times differentiable function on \([-1, 1]\), such that
\[
f(-1) = 0, f(0) = 0, f(1) = 1, f'(0) = 0.
\]
Prove that \( f^{(3)}(x) \geq 3 \) for some \( x \in (-1, 1) \).
Note that equality holds for \( \frac{1}{2}(x^3 + x^2) \).
**Hint:** Use Theorem 5.15, with $\alpha = 0$ and $\beta = 1, -1$, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.$$ 

**Proof:** By Theorem 5.15, we take $\alpha = 0, \beta = 1$,

$$f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6}$$

where $s \in (0, 1)$. Take $\alpha = 0, \text{and} \beta = -1$,

$$f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6}$$

where $t \in (-1, 0)$. Thus

$$1 = \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6}, s \in (0, 1) \quad (*)$$

$$0 = \frac{f''(0)}{2} - \frac{f^{(3)}(s)}{6}, s \in (-1, 0) \quad (**)$$

Equation (*) - equation (**):

$$\frac{f^{(3)}(s)}{6} + \frac{f^{(3)}(t)}{6}, s \in (0, 1), t \in (-1, 0).$$

$$f^{(3)}(s) + f^{(3)}(t) = 6, \quad s, t \in (-1, 1).$$

$$f^{(3)}(x) \geq 3 \quad \text{for some} \quad x \in (-1, 1).$$

**Theorem 5.15:** Suppose $f$ is a real function on $[a, b]$, $n$ is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let $\alpha, \beta$ be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$
Then there exists a point $x$ between $\alpha$ and $\beta$ such that
\[ f(\beta) = P(\beta) + \frac{f(n)(x)}{n!}(\beta - \alpha)^n. \]

18. Suppose $f$ is a real function on $[a, b]$, $n$ is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let $\alpha$, $\beta$, and $P$ be as in Taylor’s theorem (5.15). Define
\[ Q(t) = f(t) - f(\beta) \]
for $t \in [a, b]$, $t \neq \beta$, differentiate
\[ f(t) - f(\beta) = (t - \beta)Q(t) \]
n $-1$ times at $t = \alpha$, and derive the following version of Taylor’s theorem:
\[ f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^n. \]

19. Suppose $f$ is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients
\[ D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} \]
Prove the following statements:
(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
(b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.
(c) If $f'$ is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.
Give an example in which $f$ is differentiable in $(-1, 1)$ (but $f'$ is not continuous at 0) and in which $\alpha_n$, $\beta_n$ tend to 0 in such a way that
\[ \lim D_n \text{ exists but is different from } f'(0). \]

**Proof: For (a):**

\[ D_n = \frac{f(\beta_n) - f(0)}{\beta_n - \alpha_n} \frac{\beta_n}{\beta_n - \alpha_n} + \frac{f(\alpha_n) - f(0)}{\alpha_n} \frac{-\alpha_n}{\beta_n - \alpha_n} \]

Note that

\[ f'(0) = \lim_{n \to \infty} \frac{f(\alpha_n) - f(0)}{\alpha_n} = \lim_{n \to \infty} \frac{f(\beta_n) - f(0)}{\beta_n} \]

Thus for any \( \epsilon > 0 \), there exists \( N \) such that

\[ L - \epsilon < \frac{f(\alpha_n) - f(0)}{\alpha_n} < L + \epsilon, \]

\[ L - \epsilon < \frac{f(\beta_n) - f(0)}{\beta_n} < L + \epsilon, \]

whenever \( n > N \) where \( L = f'(0) \) respectively. Note that \( \beta_n/(\beta_n - \alpha_n) \) and \(-\alpha_n/(\beta_n - \alpha_n)\) are positive. Hence,

\[ \frac{\beta_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\beta_n) - f(0)}{\beta_n} \frac{\beta_n}{\beta_n - \alpha_n} < \frac{\beta_n}{\beta_n - \alpha_n}(L + \epsilon) \]

\[ \frac{-\alpha_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\alpha_n) - f(0)}{\alpha_n} \frac{-\alpha_n}{\beta_n - \alpha_n} < \frac{-\alpha_n}{\beta_n - \alpha_n}(L + \epsilon) \]

Combine two inequations,

\[ L - \epsilon < D_n < L + \epsilon \]

Hence, \( \lim D_n = L = f'(0) \).

**For (b):** We process as above prove, but note that \(-\alpha_n/(\beta_n - \alpha_n) < 0\).

Thus we only have the following inequations:

\[ \frac{\beta_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\beta_n) - f(0)}{\beta_n} \frac{\beta_n}{\beta_n - \alpha_n} < \frac{\beta_n}{\beta_n - \alpha_n}(L + \epsilon) \]
\[-\frac{\alpha_n}{\beta_n - \alpha_n}(L + \epsilon) < \frac{f(\alpha_n) - f(0)}{\alpha_n} < \frac{\alpha_n}{\beta_n - \alpha_n}(L - \epsilon)\]

Combine them:

\[L - \frac{\beta_n + \alpha_n}{\beta_n - \alpha_n}\epsilon < D_n < L + \frac{\beta_n + \alpha_n}{\beta_n - \alpha_n}\epsilon\]

Note that \(\{\frac{\beta_n}{\beta_n - \alpha_n}\}\) is bounded, ie,

\[|\frac{\beta_n}{\beta_n - \alpha_n}| \leq M\]

for some constant \(M\). Thus

\[|\frac{\beta_n + \alpha_n}{\beta_n - \alpha_n}| = |\frac{2\beta_n}{\beta_n - \alpha_n} - 1| \leq 2M + 1\]

Hence,

\[L - (2M + 1)\epsilon < D_n < L + (2M + 1)\epsilon\]

Hence, \(\lim D_n = L = f'(0)\).

**For (c):** By using Mean-Value Theorem,

\[D_n = f'(t_n)\]

where \(t_n\) is between \(\alpha_n\) and \(\beta_n\). Note that

\[\min\{\alpha_n, \beta_n\} < t_n < \max\{\alpha_n, \beta_n\}\]

and

\[\max\{\alpha_n, \beta_n\} = \frac{1}{2}(\alpha_n + \beta_n + |\alpha_n - \beta_n|)\]

\[\min\{\alpha_n, \beta_n\} = \frac{1}{2}(\alpha_n + \beta_n - |\alpha_n - \beta_n|)\]

Thus, \(\max\{\alpha_n, \beta_n\} \to 0\) and \(\min\{\alpha_n, \beta_n\} \to 0\) as \(\alpha_n \to 0\) and \(\beta_n \to 0\). By squeezing principle for limits, \(t_n \to 0\). With the continuity of \(f'\), we have

\[\lim D_n = \lim f'(t_n) = f'(\lim t_n) = f'(0)\].
Example: Let $f$ be defined by

$$f(x) = \begin{cases} 
  x^2 \sin(1/x) & (x \neq 0), \\
  0 & (x = 0).
\end{cases}$$

Thus $f'(x)$ is not continuous at $x = 0$, and $f'(0) = 0$. Take $\alpha_n = \frac{1}{\pi/2 + 2n\pi}$ and $\beta_n = \frac{1}{2n\pi}$. It is clear that $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Also,

$$D_n = \frac{-4n\pi}{\pi(\pi/2 + 2n\pi)} \to -\frac{2}{\pi}$$

as $n \to \infty$. Thus, $\lim D_n = -2/\pi$ exists and is different from $0 = f'(0)$.

20.

21.

22. Suppose $f$ is a real function on $(-\infty, \infty)$. Call $x$ a fixed point of $f$ if $f(x) = x$.

(a) If $f$ is differentiable and $f'(t) \neq 1$ for every real $t$, prove that $f$ has at most one fixed point.

(b) Show that the function $f$ defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real $t$.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real $t$, prove that a fixed point $x$ of $f$ exists, and that $x = \lim x_n$, where $x_1$ is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, ...$
(d) Show that the process described in (c) can be visualized by the zig-zag path

\[(x_1, x_2) \rightarrow (x_2, x_2) \rightarrow (x_2, x_3) \rightarrow (x_3, x_3) \rightarrow (x_3, x_4) \rightarrow \ldots\]

**Proof:** For (a): If not, then there exists two distinct fixed points, say \(x\) and \(y\), of \(f\). Thus \(f(x) = x\) and \(f(y) = y\). Since \(f\) is differentiable, by applying Mean-Value Theorem we know that

\[f(x) - f(y) = f'(t)(x - y)\]

where \(t\) is between \(x\) and \(y\). Since \(x \neq y\), \(f'(t) = 1\). This contradicts.

For (b): We show that \(0 < f'(t) < 1\) for all real \(t\) first:

\[f'(t) = 1 + (-1)(1 + e^t)^{-2}e^t = 1 - \frac{e^t}{(1 + e^t)^2}\]

Since \(e^t > 0\)

\[(1 + e^t)^2 = (1 + e^t)(1 + e^t) > 1(1 + e^t) = 1 + e^t > e^t > 0\]

for all real \(t\), thus

\[(1 + e^t)^{-2}e^t > 0\]

\[(1 + e^t)^{-2}e^t < 1\]

for all real \(t\). Hence \(0 < f'(t) < 1\) for all real \(t\).

Next, since \(f(t) - t = (1 - e^t)^{-1} > 0\) for all real \(t\), \(f(t)\) has no fixed point.

For (c): Suppose \(x_{n+1} \neq x_n\) for all \(n\). (If \(x_{n+1} = x_n\), then \(x_n = x_{n+1} = \ldots\) and \(x_n\) is a fixed point of \(f\)).

By Mean-Value Theorem,

\[f(x_{n+1}) - f(x_n) = f'(t_n)(x_{n+1} - x_n)\]
where $t_n$ is between $x_n$ and $x_{n+1}$. Thus,

$$|f(x_{n+1}) - f(x_n)| = |f'(t_n)|(x_{n+1} - x_n)$$

Note that $|f'(t_n)|$ is bounded by $A < 1$, $f(x_n) = x_{n+1}$, and $f(x_{n+1}) = x_{n+2}$. Thus

$$|x_{n+2} - x_{n+1}| \leq A|x_{n+1} - x_n|$$

$$|x_{n+1} - x_n| \leq CA^{n-1}$$

where $C = |x_2 - x_1|$. For two positive integers $p > q$,

$$|x_p - x_q| \leq |x_p - x_{p-1}| + \ldots + |x_{q+1} - x_q|$$

$$= C(A^{q-1} + A^{q-2} + \ldots + A^{p-2})$$

$$\leq \frac{CA^{q-1}}{1 - A}.$$ 

Hence

$$|x_p - x_q| \leq \frac{CA^{q-1}}{1 - A}.$$ 

Hence, for any $\epsilon > 0$, there exists $N = \lceil \log_A \frac{\epsilon}{C(1-A)} \rceil + 2$ such that $|x_p - x_q| < \epsilon$ whenever $p > q \geq N$. By Cauchy criterion we know that $\{x_n\}$ converges to $x$. Thus,

$$\lim_{n \to \infty} x_{n+1} = f(\lim_{n \to \infty} x_n)$$

since $f$ is continuous. Thus,

$$x = f(x).$$

$x$ is a fixed point of $f$.

**For (d):** Since $x_{n+1} = f(x_n)$, it is trivial.
25. Suppose \( f \) is twice differentiable on \([a, b]\), \( f(a) < 0 \), \( f(b) > 0 \), \( f'(x) \geq \delta > 0 \), and \( 0 \leq f''(x) \leq M \) for all \( x \in [a, b] \). Let \( \xi \) be the unique point in \((a, b)\) at which \( f(\xi) = 0 \).

Complete the details in the following outline of **Newton’s method** for computing \( \xi \).

(a) Choose \( x_1 \in (\xi, b) \), and define \( \{x_n\} \) by

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

Interpret this geometrically, in terms of a tangent to the graph of \( f \).

(b) Prove that \( x_{n+1} < x_n \) and that

\[
\lim_{n \to \infty} x_n = \xi.
\]

(c) Use Taylor’s theorem to show that

\[
x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2
\]

for some \( t_n \in (\xi, x_n) \).

(d) If \( A = \frac{M}{2\delta} \), deduce that

\[
0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.
\]

(Compare with Exercise 16 and 18, Chap. 3)

(e) Show that Newton’s method amounts to finding a fixed point of the function \( g \) defined by

\[
g(x) = x - \frac{f(x)}{f'(x)}.
\]

How does \( g'(x) \) behave for \( x \) near \( \xi \)?

(f) Put \( f(x) = x^{1/3} \) on \(( -\infty, \infty)\) and try Newton’s method. What happens?
Proof: For (a): You can see the picture in the following URL: http://archives.math.utk.edu/visual.calculus/3/newton.5/1.html.

For (b): We show that $x_n \geq x_{n+1} \geq \xi$. (induction). By Mean-Value Theorem, $f(x_n) - f(\xi) = f'(c_n)(x_n - \xi)$ where $c_n \in (\xi, x_n)$. Since $f'' \geq 0$, $f'$ is increasing and thus

$$\frac{f(x_n)}{x_n - \xi} = f'(c_n) \leq f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

$$f(x_n)(x_n - \xi) \leq f(x_n)(x_n - x_{n+1})$$

Note that $f(x_n) > f(\xi) = 0$ since $f' \geq \delta > 0$ and $f$ is strictly increasing. Thus,

$$x_n - \xi \leq x_n - x_{n+1}$$

$$\xi \leq x_{n+1}$$

Note that $f(x_n) > 0$ and $f'(x_n) > 0$. Thus $x_{n+1} < x_n$. Hence,

$$x_n > x_{n+1} \geq \xi.$$ 

Thus, $\{x_n\}$ converges to a real number $\zeta$. Suppose $\zeta \neq \xi$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note that $\frac{f(x_n)}{f'(x_n)} > \frac{f(\xi)}{\delta}$. Let $\alpha = \frac{f(\xi)}{\delta} > 0$, be a constant. Thus,

$$x_{n+1} < x_n - \alpha$$

for all $n$. Thus, $x_n < x_1 - (n - 1)\alpha$, that is, $x_n \to -\infty$ as $n \to \infty$. It contradicts. Thus, $\{x_n\}$ converges to $\xi$.

For (c): By using Taylor’s theorem,

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2(x_n - \xi)^2}$$

$$0 = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2(x_n - \xi)^2}$$
\[0 = \frac{f(x_n)}{f'(x_n)} - x_n + \xi + \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2\]
\[x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2\]

where \(t_n \in (\xi, x_n)\).

**For (d):** By (b) we know that \(0 \leq x_{n+1} - \xi\) for all \(n\). Next by (c) we know that
\[x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2\]

Note that \(f'' \leq M\) and \(f' \geq \delta > 0\). Thus
\[x_{n+1} - \xi \leq A(x_n - \xi)^2 \leq \frac{1}{A}(A(x_1 - \xi))^2n\]

by the induction. Thus,
\[0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^2n.\]

**For (e):** If \(x_0\) is a fixed point of \(g(x)\), then \(g(x_0) = x_0\), that is,
\[x_0 - \frac{f(x_0)}{f'(x_0)} = x_0\]
\[f(x_0) = 0.\]

It implies that \(x_0 = \xi\) and \(x_0\) is unique since \(f\) is strictly increasing. Thus, we choose \(x_1 \in (\xi, b)\) and apply Newton’s method, we can find out \(\xi\). Hence we can find out \(x_0\).

Next, by calculating
\[g'(x) = \frac{f(x)f''(x)}{f'(x)^2}\]
\[0 \leq g'(x) \leq f(x)\frac{M}{\delta^2}.\]

As \(x\) near \(\xi\) from right hand side, \(g'(x)\) near \(f(\xi) = 0\).
For (f): \(x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n\) by calculating. Thus,

\[x_n = (-2)^{n-1}x_1\]

for all \(n\), thus \(\{x_n\}\) does not converges for any choice of \(x_1\), and we cannot find \(\xi\) such that \(f(\xi) = 0\) in this case.

26. Suppose \(f\) is differentiable on \([a, b]\), \(f(a) = 0\), and there is a real number \(A\) such that \(|f'(x)| \leq A|f(x)|\) on \([a, b]\). Prove that \(f(x) = 0\) for all \(x \in [a, b]\). Hint: Fix \(x_0 \in [a, b]\), let

\[M_0 = \sup |f(x)|, M_1 = \sup |f'(x)|\]

for \(a \leq x \leq x_0\). For any such \(x\),

\[|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.\]

Hence \(M_0 = 0\) if \(A(x_0 - a) < 1\). That is, \(f = 0\) on \([a, x_0]\). Proceed.

**Proof:** Suppose \(A > 0\). (If not, then \(f = 0\) on \([a, b]\) clearly.) Fix \(x_0 \in [a, b]\), let

\[M_0 = \sup |f(x)|, M_1 = \sup |f'(x)|\]

for \(a \leq x \leq x_0\). For any such \(x\),

\[f(x) - f(a) = f'(c)(x - a)\]

where \(c\) is between \(x\) and \(a\) by using Mean-Value Theorem. Thus

\[|f(x)| \leq M_1(x - a) \leq M_1(x_0 - a) \leq A(x_0 - a)M_0\]

Hence \(M_0 = 0\) if \(A(x_0 - a) < 1\). That is, \(f = 0\) on \([a, x_0]\) by taking \(x_0 = a + \frac{1}{2A}\). Repeat the above argument by replacing \(a\) with \(x_0\), and note that \(\frac{1}{2A}\) is a constant. Hence, \(f = 0\) on \([a, b]\).
27. Let \( \phi \) be a real function defined on a rectangle \( R \) in the plane, given by \( a \leq x \leq b, \alpha \leq y \leq \beta \). A solution of the initial-value problem

\[
y' = \phi(x, y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)
\]

is, by definition, a differentiable function \( f \) on \( [a, b] \) such that \( f(a) = c \), \( \alpha \leq f(x) \leq \beta \), and

\[
f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b)
\]

Prove that such a problem has at most one solution if there is a constant \( A \) such that

\[
|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|
\]

whenever \((x, y_1) \in R\) and \((x, y_2) \in R\).

**Hint:** Apply Exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

\[
y' = y^{1/2}, \quad y(0) = 0
\]

which has two solutions: \( f(x) = 0 \) and \( f(x) = x^2/4 \). Find all other solutions.

**Proof:** Suppose \( y_1 \) and \( y_2 \) are solutions of that problem. Since

\[
|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|
\]

\( y(a) = c, \ y'_1 = \phi(x, y_1), \) and \( y'_2 = \phi(x, y_2) \), by Exercise 26 we know that \( y_1 - y_2 = 0, \ y_1 = y_2 \). Hence, such a problem has at most one solution.

**Note:** Suppose there is initial-value problem

\[
y' = y^{1/2}, \ y(0) = 0.
\]
If \( y^{1/2} \neq 0 \), then \( y^{1/2} dy = dx \). By integrating each side and noting that \( y(0) = 0 \), we know that \( f(x) = x^2/4 \). With \( y^{1/2} = 0 \), or \( y = 0 \). All solutions of that problem are \( f(x) = 0 \) and \( f(x) = x^2/4 \).

Why the uniqueness theorem does not hold for this problem? One reason is that there does not exist a constant \( A \) satisfying

\[ |y'_1 - y'_2| \leq A |y_1 - y_2| \]

if \( y_1 \) and \( y_2 \) are solutions of that problem. (since \( 2/x \to \infty \) as \( x \to 0 \) and thus \( A \) does not exist).

28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

\[ y'_j = \phi_j(x, y_1, \ldots, y_k), \quad y_j(a) = c_j \quad (j = 1, \ldots, k) \]

Note that this can be rewritten in the form

\[ \mathbf{y}' = \phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c} \]

where \( \mathbf{y} = (y_1, \ldots, y_k) \) ranges over a \( k \)-cell, \( \phi \) is the mapping of a \((k+1)\)-cell into the Euclidean \( k \)-space whose components are the function \( \phi_1, \ldots, \phi_k \), and \( \mathbf{c} \) is the vector \((c_1, \ldots, c_k)\). Use Exercise 26, for vector-valued functions.

**Theorem:** Let \( \phi_j(j = 1, \ldots, k) \) be real functions defined on a rectangle \( R_j \) in the plane given by \( a \leq x \leq b, \alpha_j \leq y_j \leq \beta_j \).

A **solution** of the initial-value problem

\[ y'_j = \phi(x, y_j), \quad y_j(a) = c_j \quad (\alpha_j \leq c_j \leq \beta_j) \]
is, by definition, a differentiable function $f_j$ on $[a, b]$ such that $f_j(a) = c_j$, $\alpha_j \leq f_j(x) \leq \beta_j$, and

$$f_j'(x) = \phi_j(x, f_j(x)) \quad (a \leq x \leq b)$$

Then this problem has at most one solution if there is a constant $A$ such that

$$|\phi_j(x, y_{j2}) - \phi_j(x, y_{j1})| \leq A|y_{j2} - y_{j1}|$$

whenever $(x, y_{j1}) \in R_j$ and $(x, y_{j2}) \in R_j$.

**Proof:** Suppose $y_1$ and $y_2$ are solutions of that problem. For each components of $y_1$ and $y_2$, say $y_{1j}$ and $y_{2j}$ respectively, $y_{1j} = y_{2j}$ by using Exercise 26. Thus, $y_1 = y_2$.

29. Specialize Exercise 28 by considering the system

$$y'_j = y_{j+1} \quad (j = 1, ..., k - 1),$$

$$y'_k = f(x) - \sum_{j=1}^{k} g_j(x)y_j$$

where $f, g_1, ..., g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + ... + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, y'(a) = c_1, ..., y^{(k-1)}(a) = c_k.$$

**Theorem:** Let $R_j$ be a rectangle in the plain, given by $a \leq x \leq b$, $\min y_j \leq y_j \leq \max y_j$. (since $y_j$ is continuous on the compact set, say
\[ a, b \], we know that \( y_j \) attains minimal and maximal.) If there is a constant \( A \) such that
\[
\begin{align*}
|y_{j+1,1} - y_{j+1,2}| &\leq A|y_{j,1} - y_{j,2}| \\
|\sum_{j=1}^{k} g_j(x)(y_{j,1} - y_{j,2})| &\leq A|y_{k,1} - y_{k,2}|
\end{align*}
\]
whenever \((x, y_{j,1}) \in R_j\) and \((x, y_{j,2}) \in R_j\).

**Proof:** Since the system \( y_1', ..., y_k' \) with initial conditions satisfies a fact that there is a constant \( A \) such that \(|y_1' - y_2'| \leq A|y_1 - y_2|\), that system has at most one solution. Hence,
\[
y^{(k)} + g_k(x)y^{(k-1)} + ... + g_2(x)y' + g_1(x)y = f(x),
\]
with initial conditions has at most one solution.