1. Prove that the convergence of \( \{s_n\} \) implies convergence of \( \{|s_n|\} \). Is the converse true?

**Solution:** Since \( \{s_n\} \) is convergent, for any \( \epsilon > 0 \), there exists \( N \) such that \( |s_n - s| < \epsilon \) whenever \( n \geq N \). By Exercise 1.13 I know that \( ||s_n| - |s|| \leq |s_n - s| \). Thus, \( ||s_n| - |s|| < \epsilon \), that is, \( \{s_n\} \) is convergent.

The converse is not true. Consider \( s_n = (-1)^n \).

2. Calculate \( \lim_{n \to \infty} (\sqrt{n^2 + n} - n) \).

**Solution:**

\[
\begin{align*}
\sqrt{n^2 + n} - n &= \frac{n}{\sqrt{n^2 + n + n} - 1} \\
&= \frac{1}{\sqrt{1/n + 1} + 1} \\
&\to \frac{1}{2}
\end{align*}
\]

as \( n \to \infty \).

3. If \( s_n = \sqrt{2} \) and

\[ s_{n+1} = \sqrt{2 + s_n} \quad (n = 1, 2, 3, \ldots), \]

prove that \( \{s_n\} \) converges, and that \( s_n < 2 \) for \( n = 1, 2, 3, \ldots \).

**Proof:** First, I show that \( \{s_n\} \) is strictly increasing. It is trivial that \( s_2 = \sqrt{2 + \sqrt{2}} = \sqrt{2 + \sqrt{2}} > \sqrt{2} = s_1 \). Suppose \( s_k > s_{k-1} \) when
The induction hypothesis yields:

\[ s_n = \sqrt{2 + \sqrt{s_{n-1}}} > \sqrt{2 + \sqrt{s_{n-2}}} = s_{n-1} \]

By the induction, \( \{s_n\} \) is strictly increasing. Next, I show that \( \{s_n\} \) is bounded by 2. Similarly, I apply the induction again. Hence \( \{s_n\} \) is strictly increasing and bounded, that is, \( \{s_n\} \) converges.

4.

5.

6.

7. Prove that the convergence of \( \sum a_n \) implies the convergence of

\[ \sum \frac{\sqrt{a_n}}{n} \]

if \( a_n \geq 0 \).

**Proof:** By Cauchy’s inequality,

\[ \sum_{n=1}^{k} a_n \frac{1}{n} \geq \sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} \]

for all \( n \in \mathbb{N} \). Also, both \( \sum a_n \) and \( \sum \frac{1}{n} \) are convergent; thus \( \sum_{n=1}^{k} a_n \frac{\sqrt{a_n}}{n} \) is bounded. Besides, \( \frac{\sqrt{a_n}}{n} \geq 0 \) for all \( n \). Hence \( \sum \frac{\sqrt{a_n}}{n} \) is convergent.

8.

9. Find the radius of convergence of each of the following power series:

(\( a \)) \( \sum n^3 z^n \), (\( b \)) \( \sum \frac{2^n}{n!} z^n \), (\( c \)) \( \sum \frac{2^n}{n^2} z^n \), (\( d \)) \( \sum \frac{n^3}{3^n} z^n \).
Solution: (a) $\alpha_n = (n^3)^{1/n} \to 1$ as $n \to \infty$. Hence $R = 1/\alpha = 1$.
(b) $\alpha_n = (2^n/n!)^{1/n} = 2/(n!)^{1/n} \to 0$ as $n \to \infty$. Hence $R = +\infty$.
(c) $\alpha_n = (2^n/n^2)^{1/n} \to 2/1 = 2$ as $n \to \infty$. Hence $R = 1/\alpha = 1/2$.
(d) $\alpha_n = (n^3/3^n)^{1/n} \to 1/3$ as $n \to \infty$. Hence $R = 1/\alpha = 3$.

10.

11. Suppose $a_n > 0$, $s_n = a_1 + \ldots + a_n$, and $\sum a_n$ diverges.
   (a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
   (b) Prove that
   \[ \frac{a_{N+1}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}} \]
   and deduce that $\sum \frac{a_n}{s_n}$ diverges.
   (c) Prove that
   \[ \frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n} \]
   and deduce that $\sum \frac{a_n}{s_n}$ converges.
   (d) What can be said about $\sum \frac{a_n}{1+na_n}$ and $\sum \frac{a_n}{1+n^2a_n}$?

Proof of (a): Note that
\[ \frac{a_n}{1+a_n} \to 0 \iff \frac{1}{\frac{1}{a_n}+1} \to 0 \]
\[ \iff \frac{1}{a_n} \to \infty \]
\[ \iff a_n \to 0 \]
as $n \to \infty$. If $\sum \frac{a_n}{1+a_n}$ converges, then $a_n \to 0$ as $n \to \infty$. Thus for some $\epsilon' = 1$ there is an $N_1$ such that $a_n < 1$ whenever $n \geq N_1$. Since $\sum \frac{a_n}{1+a_n}$ converges, for any $\epsilon > 0$ there is an $N_2$ such that
\[ \frac{a_m}{1+a_m} + \ldots + \frac{a_n}{1+a_n} < \epsilon \]
all $n > m \geq N_2$. Take $N = \max(N_1, N_2)$. Thus

$$\epsilon > \frac{a_m}{1 + a_m} + \ldots + \frac{a_n}{1 + a_n}$$

$$> \frac{a_m}{1 + 1} + \ldots + \frac{a_n}{1 + 1}$$

$$= \frac{a_m + \ldots + a_n}{2}$$

for all $n > m \geq N$. Thus

$$a_m + \ldots + a_n < 2\epsilon$$

for all $n > m \geq N$. It is a contradiction. Hence $\sum \frac{a_n}{1 + a_n}$ diverges.

**Proof of (b):**

$$\frac{a_{N+k}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + \ldots + \frac{a_{N+k}}{s_{N+k}}$$

$$= \frac{a_{N+1} + \ldots + a_{N+k}}{s_{N+k}}$$

$$= \frac{s_{N+k} - s_N}{s_{N+k}}$$

$$= 1 - \frac{s_N}{s_{N+k}}$$

If $\sum \frac{a_n}{s_n}$ converges, for any $\epsilon > 0$ there exists $N$ such that

$$\frac{a_m}{s_m} + \ldots + \frac{a_n}{s_n} < \epsilon$$

for all $m, n$ whenever $n > m \geq N$. Fix $m = N$ and let $n = N + k$. Thus

$$\epsilon > \frac{a_m}{s_m} + \ldots + \frac{a_n}{s_n}$$

$$= \frac{a_N}{s_{N+k}} + \ldots + \frac{a_{N+k}}{s_{N+k}}$$

$$\geq 1 - \frac{s_N}{s_{N+k}}$$
for all $k \in N$. But $s_{N+k} \to \infty$ as $k \to \infty$ since $\sum a_n$ diverges and $a_n > 0$. Take $\epsilon = 1/2$ and we obtain a contradiction. Hence $\sum \frac{a_n}{s_n}$ diverges.

**Proof of (c):**

\[
\begin{aligned}
s_{n-1} \leq s_n & \iff \frac{1}{s_n^2} \leq \frac{1}{s_n s_{n-1}} \\
& \iff \frac{a_n}{s_n^2} \leq \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} \\
& \iff \frac{a_n}{s_n^2} \leq \frac{1}{s_n - 1} - \frac{1}{s_n}
\end{aligned}
\]

for all $n$.

Hence

\[
\sum_{n=2}^{k} \frac{a_n}{s_n^2} \leq \sum_{n=2}^{k} \left( \frac{1}{s_n} - \frac{1}{s_n - 1} \right)
= \frac{1}{s_1} - \frac{1}{s_n}.
\]

Note that $\frac{1}{s_n} \to 0$ as $n \to \infty$ since $\sum a_n$ diverges. Hence $\sum \frac{a_n}{s_n}$ converges.

**Proof of (d):** $\sum \frac{a_n}{1+na_n}$ may converge or diverge, and $\sum \frac{a_n}{1+n^2a_n}$ converges. To see this, we put $a_n = 1/n$. $\frac{a_n}{1+na_n} = \frac{1}{2n}$, that is, $\sum \frac{a_n}{1+na_n} = 2 \sum 1/n$ diverges. Besides, if we put

\[a_n = \frac{1}{n(\log n)^p}\]

where $p > 1$ and $n \geq 2$, then

\[
\frac{a_n}{1+na_n} = \frac{1}{n(\log n)^p((\log n)^p + 1)} \leq \frac{1}{2n(\log n)^{3p}}
\]

for large enough $n$. By Theorem 3.25 and Theorem 3.29, $\sum \frac{a_n}{1+n^2a_n}$ converges. Next,

\[
\sum \frac{a_n}{1+n^2a_n} = \sum \frac{1}{1/a_n + n^2}
\]
for all $a_n$. Note that $\sum \frac{1}{n^2}$ converges, and thus $\sum \frac{a_n}{1+n^2 a_n}$ converges.

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$ 

(a) Prove that

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

**Proof of (a):**

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > \frac{a_m + \ldots + a_n}{r_m}$$

$$= \frac{r_m - r_n}{r_m}$$

$$= 1 - \frac{r_n}{r_m}$$

if $m < n$. If $\sum \frac{a_n}{r_n}$ converges, for any $\epsilon > 0$ there exists $N$ such that

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} < \epsilon$$

for all $m, n$ whenever $n > m \geq N$. Fix $m = N$. Thus

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

$$= 1 - \frac{r_n}{r_N}$$
for all \( n > N \). But \( r_n \to 0 \) as \( n \to \infty \); thus \( \frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} \to 1 \) as \( n \to \infty \).

If we take \( \epsilon = 1/2 \), we will get a contradiction.

**Proof of (b):** Note that

\[
\begin{align*}
r_{n+1} < r_n & \iff \sqrt{r_{n+1}} < \sqrt{r_n} \\
& \iff \sqrt{r_n} + \sqrt{r_{n+1}} < 2\sqrt{r_n} \\
& \iff \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2 \\
& \iff \frac{r_n - r_{n+1}}{\sqrt{r_n} + \sqrt{r_{n+1}}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
& \iff \frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})
\end{align*}
\]

since \( a_n > 0 \) for all \( n \).

Hence,

\[
\sum_{n=1}^{k} \frac{a_n}{\sqrt{r_n}} < \sum_{n=1}^{k} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_1} - \sqrt{r_{k+1}})
\]

Note that \( r_n \to 0 \) as \( n \to \infty \). Thus \( \sum \frac{a_n}{\sqrt{r_n}} \) is bounded. Hence \( \sum \frac{a_n}{\sqrt{r_n}} \) converges.

**Note:** If we say \( \sum a_n \) converges faster than \( \sum b_n \), it means that

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0.
\]

According the above exercise, we can construct a faster convergent series from a known convergent one easily. It implies that there is no **perfect** tests to test all convergences of the series from a known convergent one.
13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Note: Given \( \sum a_n \) and \( \sum b_n \), we put \( c_n = \sum_{k=0}^{n} a_k b_{n-k} \) and call \( \sum c_n \) the Cauchy product of the two given series.

Proof: Put \( A_n = \sum_{k=0}^{n} |a_k|, B_n = \sum_{k=0}^{n} |b_k|, C_n = \sum_{k=0}^{n} |c_k| \). Then

\[
C_n = |a_0 b_0| + |a_0 b_1 + a_1 b_0| + \ldots + |a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0| \\
\leq |a_0||b_0| + (|a_0||b_1| + |a_1||b_0|) + \ldots \\
+ (|a_0||b_n| + |a_1||b_{n-1}| + \ldots + |a_n||b_0|) \\
= |a_0|B_n + |a_1|B_{n-1} + \ldots + |a_n|B_0 \\
\leq |a_0|B_n + |a_1|B_n + \ldots + |a_n|B_n \\
= (|a_0| + |a_1| + \ldots + |a_n|)B_n = A_nB_n \leq AB
\]

where \( A = \lim A_n \) and \( B = \lim B_n \). Hence \( \{C_n\} \) is bounded. Note that \( \{C_n\} \) is increasing, and thus \( C_n \) is a convergent sequence, that is, the Cauchy product of two absolutely convergent series converges absolutely.

14. If \( \{a_n\} \) is a complex sequence, define its arithmetic means \( \sigma_n \) by

\[
\sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1} (n = 0, 1, 2, \ldots).
\]

(a) If \( \lim s_n = s \), prove that \( \lim \sigma_n = s \).

(b) Construct a sequence \( \{s_n\} \) which does not converge, although \( \lim \sigma_n = 0 \).

(c) Can it happen that \( s_n > 0 \) for all \( n \) and that \( \lim \sup s_n = \infty \), although \( \lim \sigma_n = 0 \)?
(d) Put \( a_n = s_n - s_{n-1} \), for \( n \geq 1 \). Show that

\[
{s_n - \sigma_n} = \frac{1}{n + 1} \sum_{k=1}^{n} k a_k.
\]

Assume that \( \lim(na_n) = 0 \) and that \( \{\sigma_n\} \) converges. Prove that \( \{s_n\} \) converges. [This gives a converse of (a), but under the additional assumption that \( na_n \to 0 \).]

(e) Derive the last conclusion from a weaker hypothesis: Assume \( M < \infty \), \( |na_n| \leq M \) for all \( n \), and \( \lim \sigma_n = \sigma \). Prove that \( \lim s_n = \sigma \), by completing the following outline:

If \( m < n \), then

\[
s_n - \sigma_n = \frac{m + 1}{n - m} (\sigma_n - \sigma_m) + \frac{1}{n - m} \sum_{i=m+1}^{n} (s_n - s_i).
\]

For these \( i \),

\[
|s_n - s_i| \leq \frac{(n - i)M}{i + 1} \leq \frac{(n - m - 1)M}{m + 2}.
\]

Fix \( \epsilon > 0 \) and associate with each \( n \) the integer \( m \) that satisfies

\[
m \leq \frac{n - \epsilon}{1 + \epsilon} < m + 1.
\]

Then \( (m + 1)/(n - m) \leq 1/\epsilon \) and \( |s_n - s_i| < M\epsilon \). Hence

\[
\limsup_{n \to \infty} |s_n - \sigma| \leq M\epsilon.
\]

Since \( \epsilon \) was arbitrary, \( \lim s_n = \sigma \).

**Proof of (a):** The proof is straightforward. Let \( t_n = s_n - s \), \( \tau_n = \sigma_n - s \).

(Or you may suppose that \( s = 0 \).) Then

\[
\tau_n = \frac{t_0 + t_1 + \ldots + t_n}{n + 1}.
\]
Choose $M > 0$ such that $|t_n| \leq M$ for all $n$. Given $\epsilon > 0$, choose $N$ so that $n > N$ implies $|t_n| < \epsilon$. Taking $n > N$ in $\tau_n = (t_0 + t_1 + \ldots + t_n)/(n+1)$, and then

$$|\tau_n| \leq \frac{|t_0| + \ldots + |t_N|}{n+1} + \frac{|t_{N+1} + \ldots + t_n|}{n+1} < \frac{(N+1)M}{n+1} + \epsilon.$$ 

Hence, $\limsup_{n \to \infty} |\tau_n| \leq \epsilon$. Since $\epsilon$ is arbitrary, it follows that $\lim_{n \to \infty} |\tau_n| = 0$, that is, $\lim \sigma_n = s$.

**Proof of (b):** Let $s_n = (-1)^n$. Hence $|\sigma_n| \leq 1/(n+1)$, that is, $\lim \sigma_n = 0$. However, $\lim s_n$ does not exists.

**Proof of (c):** Let

$$s_n = \begin{cases} 
1, & n = 0, \\
\frac{1}{n^{1/4}} + \frac{1}{n}, & n = k^2 \text{ for some integer } k, \\
\frac{1}{n}, & \text{otherwise}.
\end{cases}$$

It is obvious that $s_n > 0$ and $\limsup s_n = \infty$. Also,

$$s_0 + \ldots + s_n = 1 + \sqrt{n} \frac{n^{1/4}}{n} = 2 + \sqrt{n} \frac{n^{1/4}}{n}.$$

That is,

$$\sigma_n = \frac{2}{n+1} + \frac{\sqrt{n} \frac{n^{1/4}}{n}}{n+1}$$

The first term $2/(n+1) \to 0$ as $n \to \infty$. Note that

$$0 \leq \frac{\sqrt{n} \frac{n^{1/4}}{n}}{n+1} < n^{1/2}n^{1/4}n^{-1} = n^{-1/4}.$$

It implies that the last term $\to 0$. Hence, $\lim \sigma_n = 0$. 

10
Proof of (d):

\[ \sum_{k=1}^{n} ka_k = \sum_{k=1}^{n} k(s_k - s_{k-1}) = \sum_{k=1}^{n} ks_k - \sum_{k=1}^{n} ks_{k-1} \]

\[ = \sum_{k=1}^{n} ks_k - \sum_{k=0}^{n-1} (k + 1)s_k \]

\[ = ns_n + \sum_{k=1}^{n-1} ks_k - \sum_{k=1}^{n-1} (k + 1)s_k - s_0 \]

\[ = ns_n - \sum_{k=1}^{n-1} s_k - s_0 = (n + 1)s_n - \sum_{k=0}^{n} s_k \]

\[ = (n + 1)(s_n - \sigma_n). \]

That is,

\[ s_n - \sigma_n = \frac{1}{n + 1} \sum_{k=1}^{n} ka_k. \]

Note that \( \{na_n\} \) is a complex sequence. By (a),

\[ \lim_{n \to \infty} \left( \frac{1}{n + 1} \sum_{k=1}^{n} ka_k \right) = \lim_{n \to \infty} na_n = 0. \]

Also, \( \lim \sigma_n = \sigma \). Hence by the previous equation, \( \lim s = \sigma \).

Proof of (e): If \( m < n \), then

\[ \sum_{i=m+1}^{n} (s_n - s_i) + (m + 1)(\sigma_n - \sigma_m) \]

\[ = (n - m)s_n - \sum_{i=m+1}^{n} s_i + (m + 1)(\sigma_n - \sigma_m) \]

\[ = (n - m)s_n - \left( \sum_{i=0}^{n} s_i - \sum_{i=0}^{m} s_i \right) + (m + 1)(\sigma_n - \sigma_m) \]

\[ = (n - m)s_n - (n + 1)\sigma_n + (m + 1)\sigma_m + (m + 1)(\sigma_n - \sigma_m) \]

\[ = (n - m)s_n - (n - m)\sigma_n. \]
Hence,

\[ s_n - \sigma_n = \frac{m + 1}{n - m} (\sigma_n - \sigma_m) + \frac{1}{n - m} \sum_{i=m+1}^{n} (s_n - s_i). \]

For these \( i \), recall \( a_n = s_n - s_{n-1} \) and \( |na_n| \leq M \) for all \( n \),

\[ |s_n - s_i| = \left| \sum_{k=i+1}^{n} a_k \right| \leq \sum_{k=i+1}^{n} |a_k| \leq \sum_{k=i+1}^{n} \frac{M}{i + 1} = \frac{(n - i)M}{i + 1} \]
\[ \leq \frac{(n - (m + 1))M}{(m + 1) + 1} = \frac{(n - m - 1)M}{m + 2}. \]

Fix \( \epsilon > 0 \) and associate with each \( n \) the integer \( m \) that satisfies

\[ m \leq \frac{n - \epsilon}{1 + \epsilon} < m + 1. \]

Thus

\[ \frac{n - m}{m + 1} \geq \epsilon \quad \text{and} \quad \frac{n - m - 1}{m + 2} < \epsilon, \]

or

\[ \frac{m + 1}{n - m} \leq \frac{1}{\epsilon} \quad \text{and} \quad |s_n - s_i| < M\epsilon. \]

Hence,

\[ |s_n - \sigma| \leq |\sigma_n - \sigma| + \frac{1}{\epsilon} (|\sigma_n - \sigma| + |\sigma_m - \sigma|) + M\epsilon. \]

Let \( n \to \infty \) and thus \( m \to \infty \) too, and thus

\[ \limsup_{n \to \infty} |s_n - \sigma| \leq M\epsilon. \]

Since \( \epsilon \) was arbitrary, \( \lim s_n = \sigma \).
16. Fix a positive number $\alpha$. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots$, by the recursion formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\epsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\alpha},$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta(\frac{\epsilon_1}{\beta})^{2^n} \quad (n = 1, 2, 3, \ldots).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < \frac{1}{10}$ and therefore $\epsilon_5 < 4 \cdot 10^{-16}, \epsilon_6 < 4 \cdot 10^{-32}$.

**Proof of (a):**

$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{\alpha}{x_n})$$

$$= \frac{1}{2}(x_n - \frac{\alpha}{x_n})$$

$$= \frac{1}{2}\frac{x_n^2 - \alpha}{x_n}$$

$$> 0$$

since $x_n > \alpha$. Hence $\{x_n\}$ decreases monotonically. Also, $\{x_n\}$ is bounded by 0; thus $\{x_n\}$ converges. Let $\lim x_n = x$. Hence

$$\lim x_{n+1} = \lim \frac{1}{2}(x_n + \frac{\alpha}{x_n}) \iff x = \frac{1}{2}(x + \frac{\alpha}{x})$$

$$\iff x^2 = \alpha.$$
Note that $x_n > 0$ for all $n$. Thus $x = \sqrt{\alpha}$. $\lim x_n = \sqrt{\alpha}$.

**Proof of (b):**

\[
x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n})
\]

\[
\Rightarrow x_{n+1} - \sqrt{\alpha} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}) - \sqrt{\alpha}
\]

\[
\Rightarrow x_{n+1} - \sqrt{\alpha} = \frac{1}{2} \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{x_n}
\]

\[
\Rightarrow x_{n+1} - \sqrt{\alpha} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n}
\]

\[
\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}.
\]

Hence

\[
\epsilon_{n+1} < \beta \left( \frac{\epsilon_1}{\beta} \right)^{2^n}
\]

where $\beta = 2\sqrt{\alpha}$ by induction.

**Proof of (c):**

\[
\frac{\epsilon_1}{\beta} = \frac{2 - \sqrt{3}}{2\sqrt{3}} = \frac{1}{2\sqrt{3}(2 + \sqrt{3})} = \frac{1}{6 + 4\sqrt{3}} < \frac{1}{10}.
\]

Thus

\[
\epsilon_5 < \beta \left( \frac{\epsilon_1}{\beta} \right)^{2^4} < 2\sqrt{3} \cdot 10^{-16} < 4 \cdot 10^{-16},
\]

\[
\epsilon_6 < \beta \left( \frac{\epsilon_1}{\beta} \right)^{2^5} < 2\sqrt{3} \cdot 10^{-32} < 4 \cdot 10^{-32}.
\]

**Note:** It is an application of Newton’s method. Let $f(x) = x^2 - \alpha$ in Exercise 5.25.
19.

20.

21.

22. Suppose $X$ is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of $X$. Prove Baire’s theorem, namely, that $\bigcap_{n=1}^{\infty} G_n$ is not empty. (In fact, it is dense in $X$.) Hint: Find a shrinking sequence of neighborhoods $E_n$ such that $\overline{E_n} \subset G_n$, and apply Exercise 21.

Proof: I’ve proved it in Chapter 2 Exercise 30.

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space $X$. Show that the sequence $\{d(p_n, q_n)\}$ converges. Hint: For any $m, n$,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if $m$ and $n$ are large.

Proof: For any $\epsilon > 0$, there exists $N$ such that $d(p_n, p_m) < \epsilon$ and $d(q_m, q_n) < \epsilon$ whenever $m, n \geq N$. Note that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

It follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < 2\epsilon.$$

Thus $\{d(p_n, q_n)\}$ is a Cauchy sequence in $X$. Hence $\{d(p_n, q_n)\}$ converges.
24. Let $X$ be a metric space. (a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in $X$ equivalent if

$$\lim_{n \to \infty} d(p_n, q_n) = 0.$$  

Prove that this is an equivalence relation.

(b) Let $X^*$ be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\triangle(P, Q) = \lim_{n \to \infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\triangle(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that $\triangle$ is a distance function in $X^*$.

(c) Prove that the resulting metric space $X^*$ is complete.

Proof of (a): Suppose there are three Cauchy sequences $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$. First, $d(p_n, p_n) = 0$ for all $n$. Hence, $d(p_n, p_n) = 0$ as $n \to \infty$. Thus it is reflexive. Next, $d(q_n, p_n) = d(p_n, q_n) \to 0$ as $n \to \infty$. Thus it is symmetric. Finally, if $d(p_n, q_n) \to 0$ as $n \to \infty$ and if $d(q_n, r_n) \to 0$ as $n \to \infty$, $d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n) \to 0 + 0 = 0$ as $n \to \infty$. Thus it is transitive. Hence this is an equivalence relation.

Proof of (b):

Proof of (c): Let $\{P_n\}$ be a Cauchy sequence in $(X^*, \triangle)$. We wish to show that there is a point $P \in X^*$ such that $\triangle(P_n, P) \to 0$ as $n \to \infty$. For each $P_n$, there is a Cauchy sequence in $X$, denoted $\{Q_kn\}$, such that $\triangle(P_n, Q_kn) \to 0$ as $k \to \infty$. Let $\epsilon_n > 0$ be a sequence tending to 0 as $n \to \infty$. From the double sequence $\{Q_kn\}$ we can extract a subsequence $Q'_n$ such that $\triangle(P_n, Q'_n) < \epsilon_n$ for all $n$. From the triangle inequality, it follows that

$$\triangle(Q'_n, Q'_m) \leq \triangle(Q'_n, P_n) + \triangle(P_n, P_m) + \triangle(P_m, Q'_m).$$  \hspace{1cm} (1)
Since \( \{P_n\} \) is a Cauchy sequence, given \( \epsilon > 0 \), there is an \( N > 0 \) such that \( \Delta(P_n, P_m) < \epsilon \) for \( m, n > N \). We choose \( m \) and \( n \) so large that \( \epsilon_m < \epsilon, \epsilon_n < \epsilon \). Thus (1) shows that \( \{Q'_n\} \) is a Cauchy sequence in \( X \). Let \( P \) be the corresponding equivalence class in \( S \). Since

\[
\Delta(P, P_n) \leq \Delta(P, Q'_n) + \Delta(Q'_n, P_n) < 2\epsilon
\]

for \( n > N \), we conclude that \( P_n \to P \) as \( n \to \infty \). That is, \( X^* \) is complete.